King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Math 673 Comprehensive Exam
The First Semester of 2022-2023 (221)
Time Allowed: 180 Minutes

Name: ______________________ ID#: ______________

- Mobiles and smart devices are not allowed in this exam.
- Calculator is allowed.
- Write neatly and legibly. You may lose points for messy work.
- Show all your work. No points for answers **without justification.**

Question #	Marks	Maximum Marks
1		10
2		20
3		18
4		20
5		12
6		10
7		10
Total		100

Q:1 (10 points) Define Nystrom method and setup numerical solution formula to find numerical solution of the nonlinear integral equation

$$u(x) - \lambda \int_{0}^{x} K(x,t)u^2(t)dt = f(x)$$

using the Nystrom method with Trapezoidal rule.

Q:2 (14+6 points) Consider the integral equation $u(x) - \lambda \int_{a}^{x} K(x,t)u(t)\,dt = f(x)$, $a \le x \le b$

(a) Define an approximate solution $u_n(x)$ using Nystrom method for a general n. Write the system of equations and put it into matrix form.

(b) For $n = 3$, compute the matrix entries with $f(x) = 1$, $K(x,t) = xt^2$, $\lambda = 1/2$ and $0 \le x \le 1$.

Note: Take nodes as $x_i = a + (i-1)h$ with $h = \dfrac{b-a}{n}$, $i = 1, 2, \cdots, n+1$

Q:3 (5+3+5+5 points) Consider the integral equation

$$\lambda x(t) - \int_{a}^{b} K(t,s)x(s)\,ds = f(t)$$

(a) Define an approximate solution $u_n(x)$ using projection method for a general basis set $\{\phi_1, \phi_2, \cdots, \phi_n\}$.

(b) Define Chebyshev polynomials on the interval $(-1, 1)$.

(c) Define Collocation method using Chebyshev polynomials. Write system of equations to find unknowns.

(d) Define Galerkin method using Chebyshev polynomials. Write system of equations to find unknowns.

Q:4 (3+3+7+7 points) Consider the integral equation $5u(x) - \int_{0}^{1} e^{xt}u(t)\,dt = f(x)$

(a) State Fredholm Alternative theorem.

(b) Define degenerate kernel $K_n(x,t)$ for $K(x,t)$ using Taylor series and write the conditions that degenerate functions $g_i(x)$ and $h_i(t)$ must satisfy.

(c) If $\mathcal{K}u(x) = \int_{0}^{1} K(x,t)u(t)\,dt$ and $\mathcal{K}_n u(x) = \int_{0}^{1} K_n(x,t)u(t)\,dt$. Show that $\lim_{n\to\infty} ||\mathcal{K} - \mathcal{K}_n|| = 0$.

(d) Define the solution $u_n(x)$ using degenerate kernel method for the above mentioned integral equation.

Q:5 (12 points) Consider integral operator $\mathcal{K}$ with kernel $k(x,t) = 2x^2t + 3x^3t^3$, $0 \le t \le 1$. Use Gram-Schmidt orthogonalization process with $g(t) = t^2$ to find an eigenfunction corresponding to $\lambda = 0$.

Q:6 (10 points) Solve by converting into initial vale problem

$$u''(x) = -x - \frac{1}{2}x^2 - \int_0^x (x-t)u(t)dt$$

with $u(0) = 1$ and $u'(0) = 1$.

Q:7 (10 points) Setup a predictor corrector Exponential Time Differencing (ETD) method for

$$u(t) = u_o e^{-at} + \int_0^t e^{-a(t-s)} F(u(s))ds$$

where $0 \le t \le 1$.