1. $\lim_{x \to 0} \frac{1+x}{1+e^{\frac{1}{x}}} =$

Similar to Q21/page 79
Q28/page 80

(a) DNE ______________________________ (correct)
(b) 1
(c) 0
(d) ∞
(e) 2

as $x \to 0^-$, $\frac{1}{x} \to -\infty$, $e^{1/x} \to 0$

$\therefore \lim_{x\to 0^-} \frac{1+x}{1+e^{1/x}} = 1$

as $x \to 0^+$, $\frac{1}{x} \to \infty$, $e^{1/x} \to \infty$

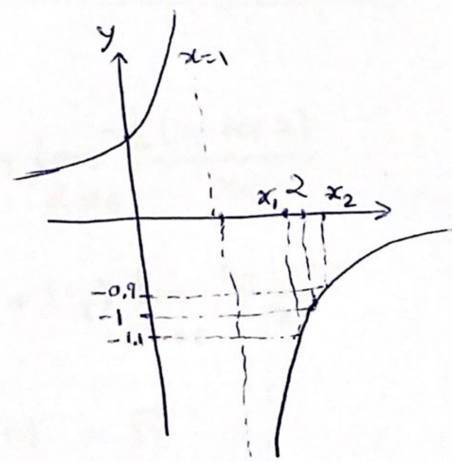
$\therefore \lim_{x\to 0^+} \frac{1+x}{1+e^{1/x}} = 0$

$\therefore \lim_{x\to 0} \frac{1+x}{1+e^{1/x}}$ DNE

2. For the function $f(x) = \dfrac{1}{1-x}$, the largest value of δ such that $|f(x)+1| < 0.1$ whenever $0 < |x-2| < \delta$ is equal to

Similar to Q # 38 page 80

(a) $\frac{1}{11}$ ______________________________ (correct)
(b) $\frac{1}{10}$
(c) $\frac{1}{12}$
(d) $\frac{1}{9}$
(e) $\frac{1}{8}$



$\because \frac{1}{1-x_1} = -1.1$

$\Rightarrow 1 - x_1 = -\frac{10}{11}$

$\Rightarrow \delta_1 = 2 - x_1 = 1 - \frac{10}{11} = \frac{1}{11}$

$\because \frac{1}{1-x_2} = -0.9$

$\Rightarrow x_2 - 1 = \frac{10}{9}$

$\Rightarrow \delta_2 = x_2 - 2 = \frac{10}{9} - 1 = \frac{1}{9}$

$\therefore$ The largest value of $\delta = \min\{\delta_1, \delta_2\} = \frac{1}{11}$

3. If $\lim_{x\to 0} \dfrac{\sqrt{2}-\sqrt{ax^2+b}}{x^2} = \dfrac{1}{\sqrt{2}}$, then $a+b =$

Similar to Q# 6 page # 118

(a) 0 (correct)

(b) -2

(c) 2

(d) 4

(e) 6

Since the given limit exist,

$\Rightarrow \lim_{x\to 0} \sqrt{2}-\sqrt{ax^2+b} = 0 \Rightarrow \sqrt{2}-\sqrt{b} = 0$
$\Rightarrow \boxed{b=2}$

Now, $\lim_{x\to 0} \dfrac{\sqrt{2}-\sqrt{ax^2+2}}{x^2} \times \dfrac{(\sqrt{2}+\sqrt{ax^2+2})}{(\sqrt{2}+\sqrt{ax^2+2})}$

$= \lim_{x\to 0} \dfrac{\cancel{2}-ax^2-\cancel{2}}{x^2(\sqrt{2}+\sqrt{ax^2+2})} = \lim_{x\to 0} \dfrac{-a}{\sqrt{2}+\sqrt{ax^2+2}} = \dfrac{-a}{2\sqrt{2}} = \dfrac{1}{\sqrt{2}}$

$\Rightarrow -\dfrac{a}{2} = 1 \Rightarrow \boxed{a=-2} \Rightarrow a+b=0$.

4. $\lim_{x\to 0} \dfrac{\sin\frac{\pi}{3}\sin 2x + \frac{1}{2}\cos x - \frac{1}{2}}{x} =$

Special Limits Page 89 + Ex. 10 / page 90

(a) $\sqrt{3}$ (correct)

(b) 1

(c) $-\sqrt{3}$

(d) -1

(e) $\sqrt{3}-1$

$= \lim_{x\to 0} \dfrac{\frac{\sqrt{3}}{2}\sin 2x}{x} + \lim_{x\to 0} \dfrac{-\frac{1}{2}(1-\cos x)}{x}$

$= \sqrt{3} \lim_{x\to 0} \dfrac{\sin 2x}{2x} + (-\tfrac{1}{2}) \lim_{x\to 0} \dfrac{1-\cos x}{x}$

$= \sqrt{3}(1) + (-\tfrac{1}{2})(0) = \sqrt{3}$

5. If $x^3 - x + 3 \le 7x + \frac{f(x)}{x} \le 3x^2 + x - 3$, then $\lim_{x\to 3} f(x) =$

Properties of limits + Squeeze Theorem

(a) 18 ________________ (correct)

(b) 27

(c) 6

(d) 21

(e) 24

$\because \lim_{x\to 3} x^3 - x + 3 = 27 - 3 + 3 = 27$

& $\lim_{x\to 3} 3x^2 + x - 3 = 3(9) + 3 - 3 = 27$

$\therefore$ By Squeeze Theorem, $\lim_{x\to 3} 7x + \frac{f(x)}{x} = 27$

$\Rightarrow 21 + \frac{\lim_{x\to 3} f(x)}{3} = 27 \Rightarrow \lim_{x\to 3} f(x) = 3(27-21) = 18$

6. The value of c for which the function

$$f(x) = \begin{cases} c^2 + cx^2, & x < 2 \\ 6c, & x = 2 \\ cx + x^3, & x > 2 \end{cases}$$

has a removable discontinuity is

Similar to Q 61–66 / page 104

(a) -4 ________________ (correct)

(b) -2

(c) 0

(d) 2

(e) 4

$\lim_{x\to 2^+} f(x) = 2c + 8$

& $\lim_{x\to 2^-} f(x) = c^2 + 4c$

for f ~~to~~ to have a removable discon. at 2, we have

$2c + 8 = c^2 + 4c \Rightarrow c^2 + 2c - 8 = 0$

$\Rightarrow (c+4)(c-2) = 0$

If $c = -4 \Rightarrow \lim_{x\to 2} f(x) = 0$ & $f(2) = -24 \Rightarrow$ f has a remov. dis.

If $c = 2 \Rightarrow \lim_{x\to 2} f(x) = 12 = f(2) \Rightarrow$ f is contin.

7. The number of discontinuities of the function $f(x) = x\,[\![x]\!]$ on the interval $\left(-\frac{5}{2}, \frac{5}{2}\right)$ is

Q # 131 / page 106.

(a) 4 ______________________________ (correct)

(b) 6

(c) 3

(d) 2

(e) 0

$[\![x]\!] = n$ if $n \leq x < n+1$

$\Rightarrow f(x) = nx$ if $n \leq x < n+1$

$\therefore$ If $n \neq 0$, $\lim_{x\to n^+} f(x) = n^2$

& $\lim_{x\to n^-} f(x) = n(n-1) = n^2 - n$

& $\because n \neq 0$ $\lim_{x\to n^+} f(x) \neq \lim_{x\to n^-} f(x)$ $\therefore f$ has a dis. at $x = n$.

if $n = 0$ $\lim_{x\to 0^+} f(x) = (0)(0) = 0$ & $\lim_{x\to 0^-} f(x) = (-1)(0) = 0$

$\therefore f$ is contin. at $x = 0$, f has discon. at $x = -2, -1, 1, 2$ in $\left(-\frac{5}{2}, \frac{5}{2}\right)$.

8. Let $f(x) = \begin{cases} a + \tan\frac{\pi x}{4} & |x| < 1 \\ bx & |x| \geq 1 \end{cases}$. If f is continuous on R, then $a^2 + b^2 =$

Similar to Q # 53 / page 104.

(a) 1 ______________________________ (correct)

(b) 4

(c) 5

(d) 8

(e) 10

f is cont. $\Rightarrow \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$

$\Rightarrow \boxed{a + 1 = b} \rightarrow ①$

& $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x)$

$\boxed{-b = a - 1} \rightarrow ②$

from ① & ②, $2a = 0 \Rightarrow \boxed{a = 0}$ & $\boxed{b = 1}$

$\therefore a^2 + b^2 = 1.$

9. The function $f(x) = \dfrac{\sin(x^2-1)}{x(x+1)}$

(a) has one vertical asymptote ______________________________ (correct)

(b) has two vertical asymptotes

(c) is continuous on R

(d) $\lim_{x\to(-1)^-} f(x) \neq \lim_{x\to(-1)^+} f(x)$

(e) $\lim_{x\to -1} f(x) = \infty$

f is contin. on $\mathbb{R}\setminus\{0,-1\}$

at $x=0$, $\lim_{x\to 0^+} \frac{\sin(x^2-1)}{x(x+1)} = -\infty$ at $x=0$ V.A.

at $x=-1$, $\lim_{x\to -1} \frac{\sin(x^2-1)}{x(x+1)} = \lim_{x\to -1} \frac{\sin(x^2-1)\,\cancel{(x-1)}}{x(x-1)(x+1)}$

$= \lim_{x\to -1} \frac{\sin(x^2-1)}{(x^2-1)} \cdot \lim_{x\to -1} \frac{x-1}{x}$

$= (1)\cdot\left(\frac{-2}{-1}\right) = 2$ f has a removable dis. No ver. asy.

10. The slope of the tangent line to the graph of $f(x) = \sin^2 x$ at $x = \frac{\pi}{4}$ is equal to

(a) $\lim_{x\to\frac{\pi}{4}} \dfrac{\sin^2 x - \frac{1}{2}}{x - \frac{\pi}{4}}$ ______________________________ (correct)

(b) $\lim_{x\to\frac{\pi}{4}} \dfrac{\sin^2 x - \frac{\sqrt{2}}{2}}{x - \frac{\pi}{4}}$

(c) $\lim_{x\to\frac{\pi}{4}} \dfrac{\sin x - \frac{\sqrt{2}}{2}}{x - \frac{\pi}{4}}$

(d) $\lim_{x\to\frac{\pi}{4}} \dfrac{\sin^2 x - \frac{1}{2}}{x^2 - \frac{\pi^2}{16}}$

(e) $\lim_{x\to\frac{\pi}{4}} \dfrac{\sin^2 x - \frac{\sqrt{2}}{2}}{x^2 - \frac{\pi}{4}}$

Def. of Tangent line with slope m & the alternative def. of the derivative.

page 121.
page 125.

11. The number of points on the graph of $f(x) = x^3 + 4$ having a tangent line parallel to the line $3x - y + 2 = 0$ is

Similar to Q#40/Page 127

(a) 2 ________________________________ (correct)

(b) 1

(c) 3

(d) 0

(e) 4

Two lines are parallel if they have the same slope. $\therefore$ slope of the tangent = slope of the line

$\Rightarrow f'(x) = 3$

$\Rightarrow 3x^2 = 3$

$\Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

$\therefore$ The points at which the tangent line is parallel to the given line are $(1, f(1)), (-1, f(-1))$ i.e. $(1,5), (-1,3)$ $\boxed{2 \text{ pts}}$

12. The value of k such that the line $y = 4x - 3$ is tangent to the graph of $f(x) = kx^2$ is

Similar to Q#72/page 190

(a) $\frac{4}{3}$ ________________________________ (correct)

(b) 2

(c) $\frac{3}{4}$

(d) $\frac{1}{2}$

(e) $\frac{1}{3}$

Let the line $y = 4x - 3$ be tangent to the graph of f at $x = a$.

$\Rightarrow f'(a) = 4$ (the slope of the line)

$\Rightarrow 2ka = 4$

$\Rightarrow \boxed{ka = 2} \rightarrow (1)$

& at a, $f(a) = 4a - 3$ (the two graphs share the same point).

$\Rightarrow ka^2 = 4a - 3$

from (1) $\Rightarrow 2a = 4a - 3 \Rightarrow \boxed{a = \frac{3}{2}}$

$\therefore k = \frac{2}{a} = 2\left(\frac{2}{3}\right) = \boxed{\frac{4}{3}}$

13. If a ball is thrown into the air with a velocity of $4\,m/s$, its height (in meters) t seconds later is given by $y = 4t - 4.9t^2$. The average velocity for the time period from $t = 1$ to $t = 3$ is

Def. of Average Velocity & Ex 10 / page 137

(a) -15.6 ____ (correct)
(b) 18.6
(c) -13.7
(d) 17.7
(e) 13.7

$$\text{Ave. velocity} = \frac{y(3) - y(1)}{3-1}$$

$$= \frac{(4(3) - (4.9)(9)) - (4 - 4.9)}{2}$$

$$= \frac{8 - (4.9)(8)}{2} = 4 - (4.9)(4)$$

$$= -15.6$$

14. A horizontal tangent line to the graph C of the function $f(x) = \dfrac{x^2}{x-1}$ at a point on C is

Q#79 / page 151

(a) $y = 4$ ____ (correct)
(b) $y = -4$
(c) $y = 2$
(d) $y = -2$
(e) $y = -1$

$f'(x) = 0$ (A hor. tang. has zero slope)

$$\Rightarrow f'(x) = \frac{2x(x-1) - x^2}{(x-1)^2}$$

$$= \frac{x^2 - 2x}{(x-1)^2} = 0$$

$$\Rightarrow x^2 - 2x = 0 \Rightarrow x(x-2) = 0 \Rightarrow x = 0, 2$$

∴ The points at which the graph has a hor. tangent line are $(0,0)$ & $(2,4)$.

∴ The hor. tangent lines are $y = 0$ & $\boxed{y = 4}$

15. $\lim_{x \to 1} \dfrac{x^3 - 1}{x^2 - 2x + 1} =$

Similar to Q # 83/page 116

(a) DNE ____ (correct)

(b) ∞

(c) $-\infty$

(d) 0

(e) 1

$$\frac{x^3-1}{x^2-2x+1} = \frac{\cancel{(x-1)}(x^2+x+1)}{(x-1)^{\cancel{2}}} = \frac{x^2+x+1}{x-1}$$

$$\therefore \lim_{x \to 1^+} \frac{x^3-1}{x^2-2x+1} = \lim_{x \to 1^+} \frac{x^2+x+1}{x-1} \begin{matrix} \to 3 \\ \to 0^+ \end{matrix} = \frac{3}{0^+} = \infty$$

$$\& \lim_{x \to 1^-} \frac{x^3-1}{x^2-2x+1} = \lim_{x \to 1^-} \frac{x^2+x+1}{x-1} \begin{matrix} \to 3 \\ \to 0^- \end{matrix} = \frac{3}{0^-} = -\infty$$

$$\therefore \lim_{x \to 1} \frac{x^3-1}{x^2-2x+1} \text{ DNE}$$

16. If $f(x) = \dfrac{1 + \cos x}{1 - \cos x}$, then $f'\left(\dfrac{\pi}{2}\right) =$

Q # 46 / page 200

(a) -2 ____ (correct)

(b) $-\sqrt{3}$

(c) 0

(d) $\sqrt{3}$

(e) 2

$$f'(x) = \frac{-\sin x(1-\cos x) - \sin x(1+\cos x)}{(1-\cos x)^2}$$

$$= \frac{-2\sin x}{(1-\cos x)^2}$$

$$f'\left(\frac{\pi}{2}\right) = \frac{(-2)(1)}{(1-0)^2} = -2$$

17. If $f(x) = -xe^x$, then $f^{(51)}(1) =$

Similar to Q# 132 / page 153

(a) $-52\,e$ ______________________________ (correct)

(b) $-50\,e$

(c) $-51\,e^2$

(d) $-52\,e^2$

(e) $-51\,e$

$f'(x) = -e^x - xe^x$

$f''(x) = -e^x - e^x - xe^x = -2e^x - xe^x$

$f'''(x) = -2e^x - e^x - xe^x = -3e^x - xe^x$

$\vdots$

$f^{(n)}(x) = -ne^x - xe^x$

$\therefore f^{(51)}(1) = -51e^1 - e^1 = -52e$

18. Which of the following statements is always true.

(a) $\lim\limits_{x\to c} f(x) = L$ means that for any given positive number q we can find a positive number p such that $|f(x) - L| < q$ whenever $0 < |x - c| < p$ ______ (correct)

(b) If f and g are functions such that g is continuous at c, then $f(g(x))$ is continuous at c

(c) If f and g are functions such that $\lim\limits_{x\to c} g(x) = L$, then $\lim\limits_{x\to c} f(g(x)) = f(L)$

(d) If f is continuous on (a, b), $f(a) \neq f(b)$ and k is any number between $f(a)$ and $f(b)$, then there exists a number $c \in (a, b)$ such that $f(c) = k$

(e) If f is a function defined on $[a, b]$, $f(a) \neq f(b)$ and k is any number between $f(a)$ and $f(b)$, then there exists a number $c \in (a, b)$ such that $f(c) = k$