1. If the points $P_1 = (1, 1)$, $P_2 = (3, y_1)$ and $P_3 = (5, y_2)$ lie on the graph of $y = f(x)$ and the slope of the secant line through P_1 and P_2 is 2 and the slope of the secant line through P_1 and P_3 is -1, then $f(3) + f(5) =$

(a) 2 ______________________________ (correct)

(b) 1

(c) -1

(d) -2

(e) 4

From the first secant line we have

$$\frac{f(3)-f(1)}{3-1} = 2 \Rightarrow f(3) = 2(2) + f(1) = 4+1 = 5$$

From the second secant line we have

$$\frac{f(5)-f(1)}{5-1} = -1 \Rightarrow f(5) = -4+1 = -3$$

$\therefore f(3) + f(5) = 5 - 3 = 2$

2. Which one of the following statements is always true?

(a) If $\lim_{x\to a^-} f(x) = -\infty$, then $x = a$ is a vertical asymptote. ______________ (correct)

(b) If $\lim_{x\to a^+} f(x) = f(a) = \lim_{x\to a^-} f(x)$, then f is differentiable at a.

(c) If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ do not exist, then $\lim_{x\to a} f(x)g(x)$ does not exist.

(d) $\lim_{x\to 0} \dfrac{\cos x - 1}{x} = -1$.

(e) $f(x) = [\![\frac{1}{2} - x]\!]$ is continuous on $\left[-\frac{1}{2}, \frac{1}{2}\right)$.

a) From the definition of the vertical asymptote.

b) false. Let $f(x) = |x|$ $\lim_{x\to 0^+} f(x) = f(0) = \lim_{x\to 0^-} f(x)$, but $f'(0)$ DNE

c) false. Let $f(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$ & $g(x) = \begin{cases} -1, & x \geq 0 \\ 1, & x < 0 \end{cases}$

$f(x)g(x) = -1$ constant function on $\mathbb{R}$.

both ~~f and g are discontin~~ $\lim_{x\to 0} f(x)$ & $\lim_{x\to 0} g(x)$ DNE

but $\lim_{x\to 0} f(x)g(x) = \lim_{x\to 0} -1 = -1$

d) $\lim_{x\to 0} \dfrac{\cos x - 1}{x} = 0 \neq -1$ e) $f(-\frac{1}{2}) = 1 \neq \lim_{x\to -\frac{1}{2}^+} f(x) = 0 \Rightarrow f$ is not contin. at $-\frac{1}{2}$

3. If $f(x) = \begin{cases} x^2+3 & x \neq 1 \\ 2 & x = 1 \end{cases}$ then $f(1) - \lim_{x\to 1} f(x) =$

Q # 26 page 79

(a) -2 ____________________ (correct)

(b) 2

(c) 0

(d) 3

(e) does not exist

$f(1) - \lim_{x\to 1} f(x) =$

$2 - [(1)^2+3] = 2-4 = -2$

4. If $y = x+4$ is the tangent line to $f(x) = k\sqrt{x}$ at $(a, f(a))$, then $a+k =$

Q # 74 page 140

(a) 8 ____________________ (correct)

(b) 4

(c) 2

(d) 6

(e) 15

$\because f(a) = k\sqrt{a} = a+4 \rightarrow ①$

Also, $f'(a) = 1 \rightarrow ②$

$\Rightarrow \frac{k}{2\sqrt{a}} = 1 \Rightarrow 2\sqrt{a} = k$

Substitute in ① we get

$a+4 = (2\sqrt{a})\sqrt{a}$ $a > 0$

$\Rightarrow a = 4$

$\Rightarrow k = 2\sqrt{4} = 2(2) = 4$

$\therefore a+k = 4+4 = 8.$

5. If $f(x) = \dfrac{x^3 - 3x^2 - 4}{x^2}$ then $f'(1) =$

Similar to Q # 45 page 139

(a) 9 ______________________________ (correct)
(b) −7
(c) 5
(d) −5
(e) 6

$f(x) = x - 3 - 4x^{-2}$

$\therefore f'(x) = 1 + 0 + 8x^{-3}$

$\Rightarrow f'(1) = 1 + 8(1)^{-3} = 9$

6. Let

$$f(x) = \begin{cases} x^3 & x \leq 2 \\ ax^2 & x > 2 \end{cases}$$

The value of a that makes f continuous is

Q # 61 page 104

(a) 2 ______________________________ (correct)
(b) −2
(c) 1
(d) 3
(e) −3

If $x \neq 2$, f is continuous at x.

for f to be continuous at $x=2$, we must

have $\lim_{x \to 2^-} f(x) = f(2) = \lim_{x \to 2^+} f(x)$

$\Rightarrow (2)^3 = 2^3 = a(2)^2$

$\Rightarrow 4a = 8 \Rightarrow a = 2$

7. Let

$$f(x) = \begin{cases} \frac{1}{2}x + 1, & x \leq 2 \\ 3 - x, & x > 2 \end{cases}$$

then

Q # 51 page 104

(a) f has a non-removable discontinuity at $x = 2$. ______________ (correct)

(b) f has a removable discontinuity at $x = 2$.

(c) f is continuous at $x = 2$.

(d) $\lim_{x\to 2} f(x)$ exists.

(e) $\lim_{x\to 2^+} f(x) = f(2)$.

$\because \lim_{x\to 2^-} f(x) = \frac{1}{2}(2) + 1 = 2 = f(2)$

$\& \lim_{x\to 2^+} f(x) = 3 - 2 = 1$

$\Rightarrow \lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$

$\therefore$ f has a non-removable discont. at $x=2$.

8. $\lim_{x\to 2^-} \frac{|x-2|}{x-2} + \lim_{x\to 2^+} \frac{x-2}{|x-2|} =$

Q # 27 page 80

(a) 0 ______________ (correct)

(b) 2

(c) -2

(d) does not exist

(e) 1

if $x \to 2^- \Rightarrow x < 2 \Rightarrow x - 2 < 0$

$\Rightarrow |x-2| = -(x-2)$

$\Rightarrow \lim_{x\to 2^-} \frac{|x-2|}{x-2} = \lim_{x\to 2^-} \frac{-\cancel{(x-2)}}{\cancel{x-2}} = -1$

& if $x \to 2^+ \Rightarrow x > 2 \Rightarrow x - 2 > 0 \Rightarrow |x-2| = x-2$

$\Rightarrow \lim_{x\to 2^+} \frac{|x-2|}{x-2} = \lim_{x\to 2^+} \frac{x-2}{x-2} = 1$

$\therefore \lim_{x\to 2^-} \frac{|x-2|}{x-2} + \lim_{x\to 2^+} \frac{|x-2|}{x-2} = -1 + 1 = 0.$

9. $\lim_{x\to -2} \frac{x^3+8}{x^4-16} =$

similar to example # 6 page 86
also, Q # 43 page 91

(a) $-\frac{3}{8}$ ______________________________ (correct)

(b) $\frac{1}{2}$

(c) $-\frac{1}{2}$

(d) 0

(e) does not exist

$$\lim_{x\to -2} \frac{x^3+8 \to 0}{x^4-16 \to 0}$$

$$= \lim_{x\to -2} \frac{\cancel{(x+2)}(x^2-2x+4)}{(x-2)\cancel{(x+2)}(x^2+4)}$$

$$= \lim_{x\to -2} \frac{x^2-2x+4}{(x-2)(x^2+4)} = \frac{4+4+4}{(-4)(4+4)}$$

$$= -\frac{12}{32} = -\frac{3}{8}.$$

10. $\lim_{x\to 0} \frac{x-\cos x+1}{2x+\sin x} =$

Theorem 2.9 Special Limits

(a) $\frac{1}{3}$ ______________________________ (correct)

(b) $\frac{2}{3}$

(c) 1

(d) $-\frac{1}{2}$

(e) does not exist

$$\lim_{x\to 0} \frac{x-\cos x+1}{2x+\sin x}$$

$$= \lim_{x\to 0} \frac{\frac{x+1-\cos x}{x}}{\frac{2x+\sin x}{x}}$$

$$= \lim_{x\to 0} \frac{1+\frac{1-\cos x}{x}}{2+\frac{\sin x}{x}} = \frac{1+0}{2+1} = \frac{1}{3}.$$

11. $\lim_{x\to 64} \dfrac{16 - x^{\frac{1}{3}}}{16 - \sqrt{x}} =$

Application of Theorem 2.4 & 2.2 pages 83-84

(a) $\dfrac{3}{2}$ ________________________________ (correct)

(b) 0

(c) does not exist

(d) $\dfrac{1}{2}$

(e) $-\infty$

$$\lim_{x\to 64} \frac{16 - x^{1/3}}{16 - \sqrt{x}}$$

$$= \lim_{x\to 64} \frac{16 - \sqrt[3]{x}}{16 - \sqrt{x}} = \frac{16 - \sqrt[3]{64}}{16 - \sqrt{64}}$$

$$= \frac{16 - 4}{16 - 8} = \frac{12}{8} = \frac{3}{2}.$$

12. $\lim_{x\to -\infty} \left(\dfrac{2}{x+1} - \dfrac{x+1}{2} \right) =$

Similar to Q#18 page 246

(a) ∞ ________________________________ (correct)

(b) $-\infty$

(c) 0

(d) -4

(e) 4

$$\because \lim_{x\to -\infty} \frac{2}{x+1} = 0$$

$$\Rightarrow \lim_{x\to -\infty} \left(\frac{2}{x+1} - \frac{x+1}{2} \right) = \lim_{x\to -\infty} -\frac{x+1}{2} = \infty.$$

13. $\lim_{x\to-\infty} \dfrac{\sqrt{3x^6+\cos x}}{2x^3+\sin x} =$

(a) $-\dfrac{\sqrt{3}}{2}$ ______________________________ (correct)

(b) $+\infty$

(c) $-\infty$

(d) $-\sqrt{3}$

(e) $-\dfrac{\sqrt{3}}{3}$

$$\lim_{x\to-\infty} \frac{\sqrt{3x^6+\cos x}}{2x^3+\sin x} = \lim_{x\to-\infty} \frac{\sqrt{x^6(3+\cos x/x^6)}}{x^3(2+\sin x/x^3)}$$

$$= \lim_{x\to-\infty} \frac{|x^3|\sqrt{3+\cos x/x^6}}{x^3(2+\sin x/x^3)} = \lim_{x\to-\infty} -\frac{\sqrt{3+\cos x/x^6}}{2+\sin x/x^3}$$

$$= -\frac{\sqrt{3+0}}{2+0} = -\frac{\sqrt{3}}{2}.$$

(Note $\because$ $-1 \le \sin x \le 1 \Rightarrow -\frac{1}{x^3} \le \frac{\sin x}{x^3} \le \frac{1}{x^3}$ & $\because \lim_{x\to-\infty} -\frac{1}{x^3} = \lim_{x\to-\infty} \frac{1}{x^3} = 0$

by Squeeze Th. $\Rightarrow \lim_{x\to-\infty} \frac{\sin x}{x^3} = 0$ Similarly, $\lim_{x\to-\infty} \frac{\cos x}{x^6} = 0$.).

14. The number of vertical asymptotes the function $f(x) = \dfrac{x}{\tan x}$ has in the interval $[-\pi, \pi]$ is

Similar to Q#32 page 112

(a) 2 ______________________________ (correct)

(b) 0

(c) 1

(d) 3

(e) 4

$$f(x) = \frac{x}{\frac{\sin x}{\cos x}} = \frac{x\cos x}{\sin x}$$

& $\because \sin x = 0 \Rightarrow x = 0, \pm\pi$ in $[-\pi, \pi]$

& $x\cos x \ne 0$ at $x = \pm\pi$

By Th. 2.14 $f(x)$ has a vertical asymptote at $x = \pm\pi$.

& $\because \lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x}{\sin x} \cdot \lim_{x\to 0} \cos x = 1 \cdot 1 = 1$

$\therefore$ $f(x)$ has no vertical asymptote at $x = 0$.

15. $\lim_{x\to 0^+}\left(\frac{1}{x}-\sin\frac{1}{x}\right) =$

(a) $+\infty$ ________________ (correct)

(b) 0

(c) 1

(d) -1

(e) $-\infty$

$\because \ -1 \leq -\sin\frac{1}{x} \leq 1$ on $(0,1)$

$\Rightarrow \ \frac{1}{x}-1 \leq \frac{1}{x}-\sin\frac{1}{x} \leq \frac{1}{x}+1$ on $(0,1)$

$\& \because \ \lim_{x\to 0^+} \frac{1}{x}-1 = \infty = \lim_{x\to 0^+}\frac{1}{x}+1$

$\therefore$ By the Squeeze Theorem (Th. 2.8),

$\lim_{x\to 0^+} \frac{1}{x}-\sin\frac{1}{x} = \infty.$

16. The equation of the tangent line to the curve $y = \sqrt[3]{x-1}+2$ at the point $(1,2)$ is

Similar to example 7 page 126.

(a) $x = 1$ ________________ (correct)

(b) $x = 2$

(c) $y = 1$

(d) $y = 2$

(e) $y = x$

The curve is contin. at $(1,2)$.

The slope of the tangent line to the curve at $x=1$ is

$$f'(1) = \lim_{x\to 1}\frac{f(x)-f(1)}{x-1} =$$

$$= \lim_{x\to 1}\frac{\sqrt[3]{x-1}+2-2}{x-1} = \lim_{x\to 1}\frac{\sqrt[3]{x-1}}{x-1}$$

$$= \lim_{x\to 1}\frac{1}{(x-1)^{2/3}} = \infty.$$

$\therefore$ The tangent line at $(1,2)$ is vertical

$\therefore$ The eq. of the tangent line at $(1,2)$ is

$x = 1$

17. The slope of the tangent line to the curve $y = \dfrac{1}{x-1}$ at the point $\left(3, \dfrac{1}{2}\right)$ is

Similar to Q # 25 page 127

(a) $-\dfrac{1}{4}$ ______________________________ (correct)

(b) $\dfrac{1}{3}$

(c) $-\dfrac{1}{9}$

(d) $\dfrac{1}{4}$

(e) $\dfrac{1}{9}$

The slope of the tangent line is

$$f'(3) = \lim_{x \to 3} \frac{f(x) - f(3)}{x-3}$$

$$= \lim_{x\to 3} \frac{\frac{1}{x-1} - \frac{1}{2}}{x-3} = \lim_{x \to 3} \frac{2-x+1}{2(x-1)(x-3)}$$

$$= \lim_{x\to 3} \frac{-\cancel{(x-3)}}{2(x-1)\cancel{(x-3)}} = \lim_{x\to 3} \frac{-1}{2(x-1)} = -\frac{1}{4}.$$

18. The sum of all values of x in $[0, 2\pi)$ at which the function $f(x) = x + \cos x$ has a horizontal tangent line is equal to

Similar to Q # 69 page 140

(a) $\dfrac{\pi}{2}$ ______________________________ (correct)

(b) $\dfrac{3\pi}{2}$

(c) π

(d) 2π

(e) 3π

The function has a horizontal tangent line when $f'(x) = 0$

$\Rightarrow f'(x) = 1 - \sin x = 0$

$\Rightarrow \sin x = 1$ in $[0, 2\pi)$

$\Rightarrow x = \dfrac{\pi}{2}$ only in $[0, 2\pi)$

Math 101, 232, Major Exam I

Answer KEY

Q	MASTER	CODE01	CODE02	CODE03	CODE04
1	A	D 5	A 17	B 5	E 5
2	A	C 6	A 1	B 12	B 7
3	A	E 7	A 4	C 1	A 2
4	A	D 15	A 13	C 7	C 17
5	A	D 1	D 11	B 4	D 16
6	A	D 17	C 10	C 9	E 4
7	A	D 10	A 8	D 8	E 6
8	A	C 16	B 3	D 16	B 11
9	A	A 8	A 14	D 2	A 9
10	A	A 4	B 16	D 11	B 8
11	A	A 13	A 9	A 10	B 15
12	A	D 14	B 15	A 13	C 12
13	A	C 9	B 7	B 18	C 3
14	A	C 3	C 18	A 3	E 18
15	A	C 18	D 2	D 6	A 14
16	A	B 12	E 5	C 17	D 10
17	A	C 2	B 12	B 14	A 13
18	A	A 11	A 6	B 15	C 1