1. The **average value** of $f(x) = x^2 - 6x + 9$ over the interval $[0, 3]$ is equal to

(a) 3 (correct)

(b) 1

(c) 9

(d) 6

(e) 12

$$f_{ave} = \frac{1}{3-0}\int_0^3 (x^2 - 6x + 9)\,dx$$

$$= \frac{1}{3}\cdot \left.\frac{x^3}{3} - 3x^2 + 9x\right]_0^3$$

$$= \frac{1}{3}\left[(9 - 27 + 27) - 0\right]$$

$$= \frac{1}{3}\cdot 9 = 3.$$

2. The **volume** of the solid generated by rotating the region bounded by the curves

$$y = \frac{1}{x^2},\ y = 0,\ x = 1,\ x = 5$$

about the y-axis is equal to

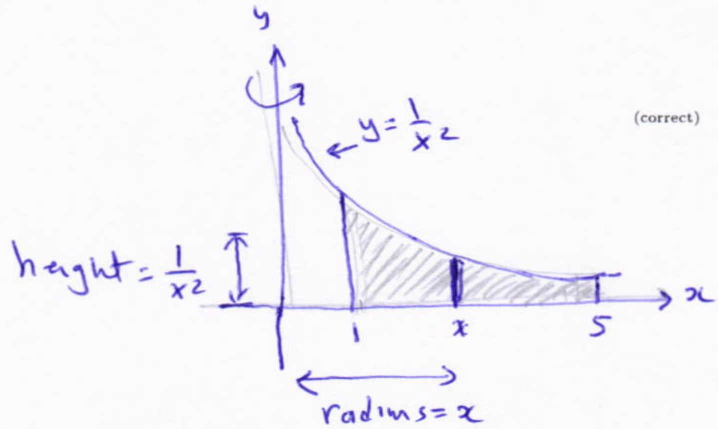
(a) $2\pi \ln 5$ (correct)

(b) $2\pi \ln 4$

(c) $2\pi \ln 3$

(d) $\pi \ln 5$

(e) $\pi \ln 4$



Shell Method

$$V = 2\pi \int_1^5 x\cdot\frac{1}{x^2}\,dx$$

$$= 2\pi \int_1^5 \frac{1}{x}\,dx$$

$$= 2\pi \left.\ln|x|\right]_1^5$$

$$= 2\pi \ln 5.$$

3. $\int 9\sqrt{x}\ln x\,dx =$ $9\int \sqrt{x}\ln x\,dx = 9\int \underbrace{\ln x}_{u} \cdot \underbrace{\sqrt{x}\,dx}_{dv}$

$u = \ln x$ $\quad dv = \sqrt{x}\,dx$

$du = \frac{1}{x}\,dx$ $\quad v = \frac{2}{3}x^{3/2}$

(a) $6x^{3/2}\ln x - 4x^{3/2} + C$ (correct)

(b) $9x^{3/2}\ln x + 2x^{3/2} + C$

(c) $2x^{3/2}\ln x - 6x^{3/2} + C$

(d) $-6x^{3/2}\ln x + 4x^{3/2} + C$

(e) $6x^{3/2}\ln x - 6x^{3/2} + C$

$= 9\left[uv - \int v\,du\right]$

$= 9\left[\frac{2}{3}x^{3/2}\ln x - \int \frac{2}{3}x^{1/2}\,dx\right.$

$= 9\left[\frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\cdot\frac{2}{3}x^{3/2}\right] + C$

$= 6x^{3/2}\ln x - 4x^{3/2} + C$

4. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^4 t\cdot\sec^4 t\,dt =$ $\int_{-\pi/4}^{\pi/4} \tan^4 t\cdot\sec^2 t\cdot\sec^2 t\,dt$

$= \int_{-\pi/4}^{\pi/4} \tan^4 t\cdot(1+\tan^2 t)\cdot\sec^2 t\,dt$

$u = \tan t \Rightarrow du = \sec^2 t\,dt$

$t = -\frac{\pi}{4} \Rightarrow u = -1$

$t = \frac{\pi}{4} \Rightarrow u = 1$

(a) $\frac{24}{35}$ (correct)

(b) $\frac{8}{15}$

(c) $\frac{16}{35}$

(d) $\frac{4}{35}$

(e) $\frac{1}{5}$

$= \int_{-1}^{1} u^4(1+u^2)\,du$

$= \int_{-1}^{1}(u^4+u^6)\,du = \left.\frac{u^5}{5}+\frac{u^7}{7}\right]_{-1}^{1}$

$= \left(\frac{1}{5}+\frac{1}{7}\right) - \left(-\frac{1}{5}-\frac{1}{7}\right) = \frac{2}{5}+\frac{2}{7} = \frac{14+10}{35} = \frac{24}{35}$

5. $\displaystyle\int_0^{\frac{\pi}{20}} \sin^2(5x)\,dx =$ $\displaystyle \frac{1}{2}\int_0^{\pi/20} [1 - \cos(10x)]\,dx$

$$= \frac{1}{2}\left[x - \frac{1}{10}\sin(10x)\right]_0^{\pi/20}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{20} - \frac{1}{10}\sin\left(\frac{\pi}{2}\right)\right) - 0\right]$$

$$= \frac{1}{2}\left(\frac{\pi}{20} - \frac{1}{10}\right)$$

$$= \frac{1}{40}(\pi - 2)$$

(a) $\dfrac{1}{40}(\pi - 2)$ (correct)

(b) $\dfrac{\pi}{40}$

(c) $\dfrac{1}{20}(\pi - 1)$

(d) $\dfrac{1}{20}(2\pi - 1)$

(e) $\dfrac{1}{40}(\pi - 1)$

6. $\displaystyle\int \frac{1}{\sqrt{9x^2+1}}\,dx =$ $\displaystyle\int \frac{1}{\sqrt{(3x)^2+1}}\,dx$. Let $3x = \tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

Then $3\,dx = \sec^2\theta\,d\theta$

$\sqrt{(3x)^2+1} = \sqrt{\tan^2\theta + 1} = \sqrt{\sec^2\theta}$

$= |\sec\theta| = \sec\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

(a) $\dfrac{1}{3}\ln|\sqrt{9x^2+1} + 3x| + C$ (correct)

(b) $\dfrac{1}{3}\ln|\sqrt{9x^2+1} + x| + C$

(c) $\ln|\sqrt{9x^2+1} - 3x| + C$

(d) $\ln|\sqrt{9x^2+1} + 6x| + C$

(e) $\dfrac{1}{3}\ln|\sqrt{9x^2+1} - 2x| + C$

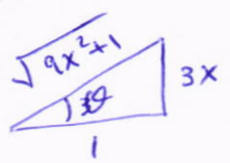
$$= \int \frac{1}{\sec\theta}\cdot\frac{1}{3}\sec^2\theta\,d\theta$$

$$= \frac{1}{3}\int \sec\theta\,d\theta$$

$$= \frac{1}{3}\ln|\sec\theta + \tan\theta| + C$$

$$= \frac{1}{3}\ln|\sqrt{9x^2+1} + 3x| + C$$

$\tan\theta = 3x$



$\sec\theta = \sqrt{9x^2+1}$

7. The improper integral $\int_0^5 \frac{1}{\sqrt[3]{5-x}}\,dx$ is

(a) convergent and its value is $\frac{3}{2}\sqrt[3]{25}$ (correct)

(b) convergent and its value is $\frac{3}{2}\sqrt[3]{5}$

(c) convergent and its value is $-\frac{3}{2}\sqrt[3]{25}$

(d) convergent and its value is $\frac{3}{4}\sqrt[3]{5}$

(e) divergent

$$= \lim_{t\to 5^-} \int_0^t (5-x)^{-1/3}\,dx$$
$$= \lim_{t\to 5^-} -\frac{3}{2}(5-x)^{2/3}\Big]_0^t$$
$$= \lim_{t\to 5^-} -\frac{3}{2}(t-5)^{2/3} + \frac{3}{2}(5-0)^{2/3}$$
$$= 0 + \frac{3}{2}5^{2/3}$$
$$= \frac{3}{2}\sqrt[3]{25}, \text{ conv.}$$

8. Let R be the region bounded by the curves $x = 2y^2$ and $x = y^2 + 1$. The **volume** of the solid generated by rotating R about the line $y = -2$ is given by

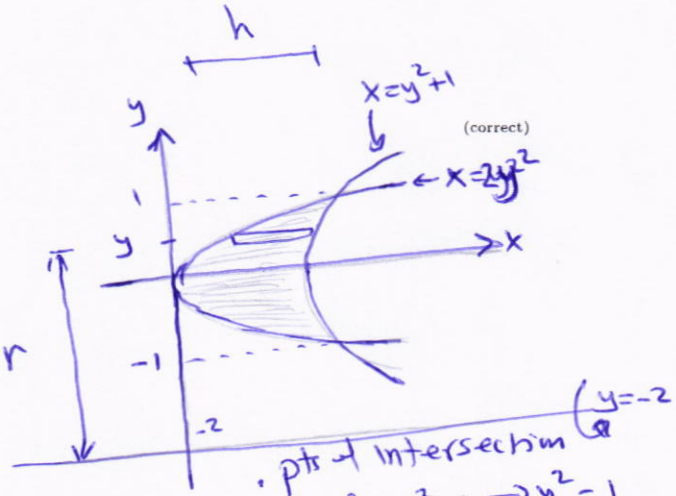
(a) $\int_{-1}^{1} 2\pi(y+2)(1-y^2)\,dy$ (correct)

(b) $\int_{-1}^{1} 2\pi(y-2)(y^2-1)\,dy$

(c) $\int_{-1}^{1} 2\pi(y-2)(1-y^2)\,dy$

(d) $\int_{-1}^{1} 2\pi y(1-y^2)\,dy$

(e) $\int_{-1}^{1} 2\pi(y+2)(y^2-1)\,dy$



• pts of intersection
$2y^2 = y^2+1 \Rightarrow y^2 = 1$
$\Rightarrow y = \pm 1$.

• $r = y-(-2) = y+2$
$h = (y^2+1) - 2y^2 = 1-y^2$

$$V = \int_{-1}^{1} 2\pi\cdot(y+2)(1-y^2)\,dy$$

9. The improper integral $\int_e^\infty \frac{1}{x(\ln x)^2}\,dx$ is

$= \lim_{t\to\infty} \int_e^t \frac{1}{x(\ln x)^2}\,dx$

$u = \ln x \Rightarrow du = \frac{1}{x}\,dx$

$x = e \Rightarrow u = 1$

$x = t \to u = \ln t$

(a) convergent and its value is 1 (correct)

(b) convergent and its value is 0

(c) convergent and its value is $\frac{1}{2}$

(d) convergent and its value is 3

(e) divergent

$= \lim_{t\to\infty} \int_1^{\ln t} \frac{1}{u^2}\,du$

$= \lim_{t\to\infty} -\frac{1}{u}\Big]_1^{\ln t}$

$= \lim_{t\to\infty} -\frac{1}{\ln t} - (-1) = 0 + 1 = 1$

conv.

10. $\int 18\sin(4x)\cos(5x)\,dx =$

$18 \int \frac{1}{2}\left[\sin(4x-5x) + \sin(4x+5x)\right] dx$

$= 9 \int [\sin(-x) + \sin(9x)]\,dx$

$= 9 \int [-\sin x + \sin(9x)]\,dx$

$= 9\left(\cos x - \frac{1}{9}\cos(9x)\right) + C$

$= 9\cos x - \cos(9x) + C$

(a) $9\cos x - \cos(9x) + C$ (correct)

(b) $18\cos x - 2\cos(9x) + C$

(c) $9\sin x - \sin(9x) + C$

(d) $9\sin x + \sin(9x) + C$

(e) $18\cos x + 2\sin(9x) + C$

11. $\int_0^{\frac{1}{2}} 8\tan^{-1}(2y)\,dy =$

$\int 8\tan^{-1}(2y)\,dy = 8\int \underbrace{\tan^{-1}(2y)}_{u}\,\underbrace{dy}_{dv}$

$u = \tan^{-1}(2y) \qquad dv = dy$

$du = \frac{2}{4y^2+1}\,dy \qquad v = y$

$= 8\left[uv - \int v\,du\right]$

$= 8\left[y\tan^{-1}(2y) - \int \frac{2y}{4y^2+1}\,dy\right]$

$= 8\left[y\tan^{-1}(2y) - \frac{1}{4}\ln(4y^2+1)\right] + C$

$= 8y\tan^{-1}(2y) - 2\ln(4y^2+1) + C$

So $\int_0^{1/2} 8\tan^{-1}(2y)\,dy = 8y\tan^{-1}(2y) - 2\ln(4y^2+1)\Big]_0^{1/2}$

$= (4\tan^{-1}(1) - 2\ln 2) - (0-0) = \pi - 2\ln 2$

(a) $\pi - 2\ln 2$ (correct)

(b) $2\pi - \ln 2$

(c) $\frac{\pi}{2} + 3\ln 2$

(d) $\frac{\pi}{4} + \frac{\ln 2}{2}$

(e) $\pi - \frac{\ln 2}{2}$

12. Using the substitution $t = \tan\left(\frac{x}{2}\right)$, $-\pi < x < \pi$,

$$\int \frac{1+\sin x}{2+\cos x}\,dx =$$

$\int \frac{1 + \frac{2t}{1+t^2}}{2 + \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt$

$= \int \frac{\frac{1+t^2+2t}{1+t^2}}{\frac{2+2t^2+1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt$

$= \int \frac{(t+1)^2}{t^2+3} \cdot \frac{2}{1+t^2}\,dt$

$= \int \frac{2(t+1)^2}{(t^2+1)(t^2+3)}\,dt$

(a) $\int \frac{2(t+1)^2}{(t^2+1)(t^2+3)}\,dt$ (correct)

(b) $\int \frac{2(t+1)^2}{(t^2+1)(t^2+2)}\,dt$

(c) $\int \frac{2t^2+t}{(t^2+1)^2}\,dt$

(d) $\int \frac{2}{t^2+3}\,dt$

(e) $\int \frac{2t^2+4t}{(t^2+1)(2t^2+1)}\,dt$

13. $\displaystyle\int_0^1 \frac{2}{2x^2+3x+1}\,dx =$

(a) $2\ln\left(\frac{3}{2}\right)$

(b) $4\ln\left(\frac{3}{2}\right)$

(c) $\ln\left(\frac{81}{4}\right)$

(d) $\ln\left(\frac{9}{8}\right)$

(e) $\ln\left(\frac{27}{8}\right)$

(correct)

$\cdot\ \dfrac{2}{(2x+1)(x+1)} = \dfrac{A}{2x+1} + \dfrac{B}{x+1}$

$\Rightarrow 2 = A(x+1) + B(2x+1)$

$\cdot\ x=-1 \Rightarrow 2 = 0 + B(-1) \Rightarrow B = -2$

$\cdot\ x = -\frac{1}{2} \Rightarrow 2 = A(\frac{1}{2}) + 0 \Rightarrow A = 4$

$\displaystyle\int_0^1 \frac{4}{2x+1} - \frac{2}{x+1}\,dx$

$= 2\ln|2x+1| - 2\ln|x+1| \Big]_0^1$

$= (2\ln 3 - 2\ln 2) - (0-0)$

$= 2(\ln 3 - \ln 2)$

$= 2\ln\left(\frac{3}{2}\right).$

14. $\displaystyle\int \frac{1}{\sqrt{x}+x\sqrt{x}}\,dx =$ $\displaystyle\int \frac{1}{\sqrt{x}(1+x)}\,dx$

$\cdot$ Let $u = \sqrt{x}$.
Then $x = u^2 \Rightarrow dx = 2u\,du$

(a) $2\tan^{-1}(\sqrt{x}) + C$

(b) $\tan^{-1}(x\sqrt{x}) + C$

(c) $\ln|1+\sqrt{x}| + C$

(d) $2\ln|1+\sqrt{x}| + C$

(e) $4\tan^{-1}(x+\sqrt{x}) + C$

(correct)

$\displaystyle = \int \frac{1}{u(1+u^2)} \cdot 2u\,du$

$\displaystyle = 2\int \frac{1}{u^2+1}\,du$

$= 2\tan^{-1} u + C$

$= 2\tan^{-1}(\sqrt{x}) + C.$

15. $\displaystyle\int \frac{\cos x}{\sin^3 x + \sin x}\, dx =$

(a) $\ln|\sin x| - \frac{1}{2}\ln(1+\sin^2 x) + C$ (correct)

(b) $\ln|\cos x| - \ln(1+\cos^2 x) + C$

(c) $\ln|\sin^3 x + \sin x| + C$

(d) $2\ln|\sin x| - \tan^{-1}(\sin x) + C$

(e) $2\ln|\sin x| - \frac{1}{2}\ln|1+\sin x| + C$

$u = \sin x$. Then $du = \cos x\, dx$

$= \int \frac{1}{u^3+u}\, du$

$\cdot \frac{1}{u(u^2+1)} = \frac{A}{u} + \frac{Bu+C}{u^2+1}$

$\Rightarrow 1 = A(u^2+1) + u(Bu+C)$

$= Au^2 + A + Bu^2 + Cu$

$= (A+B)u^2 + Cu + A$

$\Rightarrow \left.\begin{array}{l} A+B=0 \\ C=0 \\ A=1 \end{array}\right\} \Rightarrow \begin{array}{l} A=1 \\ B=-1 \\ C=0 \end{array}$

$= \int \frac{1}{u} - \frac{u}{u^2+1}\, du$

$= \ln|u| - \frac{1}{2}\ln(u^2+1) + C$

$= \ln|\sin x| - \frac{1}{2}\ln(\sin^2 x + 1) + C$