$\boxed{\text{KEY}}$

1. The statement that is **wrong** about the parametric curve

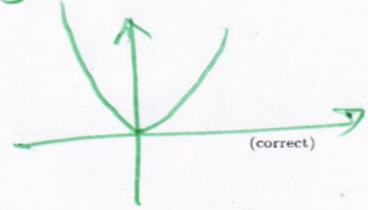
$$x = \sinh(t),\ y = -1 + \cosh(t),\ t \in (-\infty, \infty)$$

is

$\cosh^2(t) - \sinh^2(t) = 1$

$(y+1)^2 - x^2 = 1$

$y^2 + 2y - x^2 = 0.$



(a) The curve is the lower part of a hyperbola. (correct)

(b) The curve passes through the origin.

(c) The curve is symmetric about the y-axis.

(d) The cartesian equation of the curve is $x^2 - y^2 - 2y = 0$.

(e) For $t < 0$, the curve lies in the second quadrant.

$\cosh(t) \geq 1$

$-1 + \cosh(t) \geq 0$

$y \geq 0.$

Upper branch of the hyperbola $(y+1)^2 - x^2 = 1$ with vertex $(0,0)$ and openning upwards.

2. The surface area of the solid obtained by rotating the parametric curve

$$x(t) = t^3 - t,\ y = t^3 - t,\ t \in [2, 3]$$

about the x-axis is

$$S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi \int_\alpha^\beta y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

(a) $540\sqrt{2}\,\pi$ (correct)

(b) $270\sqrt{3}\,\pi$

(c) $150\sqrt{3}\,\pi$

(d) $270\sqrt{5}\,\pi$

(e) $450\sqrt{2}\,\pi$

$\frac{dx}{dt} = 3t^2 - 1$

$\frac{dy}{dt} = 3t^2 - 1$

$$= 2\pi \int_2^3 (t^3 - t)\sqrt{(3t^2-1)^2 + (3t^2-1)^2}\, dt$$

$$= 2\pi\sqrt{2} \int_2^3 \big((t^3 - t)(3t^2 - 1)\big)\, dt$$

$$= 2\sqrt{2}\pi \int_2^3 (3t^5 - 4t^3 + t)\, dt$$

$$= 2\sqrt{2}\pi \left(\frac{t^6}{2} - t^4 + \frac{t^2}{2}\right)\Big|_2^3$$

$$= 2\sqrt{2}\pi\,(270) = 540\sqrt{2}\pi.$$

10.3

3. The number of points of intersection between the polar curves

$$r = \sin(2\theta) \text{ and } r = \frac{1}{3}$$

is

rose with 4 leaves with tips at $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

circle centered at the origin with radius $\frac{1}{3}$.

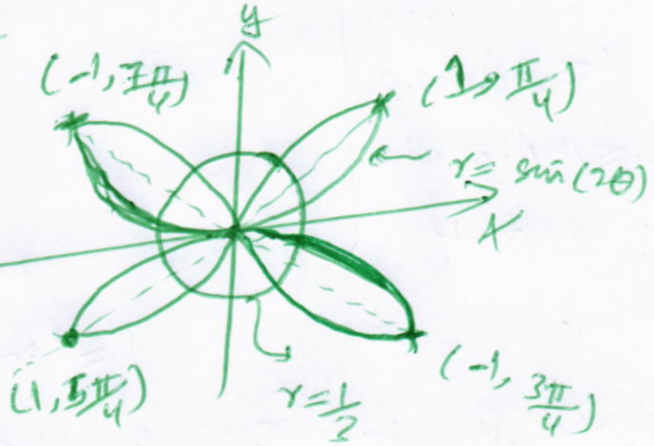
(a) 8 (correct)

(b) 6

(c) 4

(d) 2

(e) 0



The graphs of $r = \frac{1}{3}$ & $r = \sin(2\theta)$ is similar to the one given in Example 3 of 10.4

4. The graph of $r^2 = \sin(2\theta)$ is symmetric about

Since $r^2 \geq 0$; $0 + 2k\pi \leq \theta \leq \frac{\pi}{2} + 2k\pi$, $k \in \mathbb{Z}$

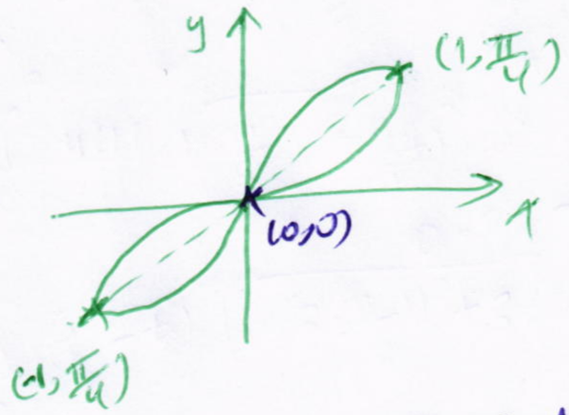
(a) the origin only. (correct)

(b) the origin and the polar axis.

(c) the x-axis and the y-axis.

(d) the origin and the line $\theta = \frac{\pi}{2}$.

(e) the polar axis only.



The graph is given in Exercise 5 of 10.4.

Symmetry about the origin $(-r, \theta)$: $(-r)^2 = \sin(2\theta)$.

From 10.3

5. The area of the region that lies **inside** the cardioid $r = 1 - \sin(\theta)$ and **outside** the circle is $r = 1$ is

Exercise 25 of 10.4

$A = \frac{1}{2}\int_\alpha^\beta (r_{out}^2 - r_{in}^2)\, d\theta$

$A = \frac{1}{2}\int_\pi^{2\pi} ((1-\sin\theta)^2 - (1)^2)\, d\theta$

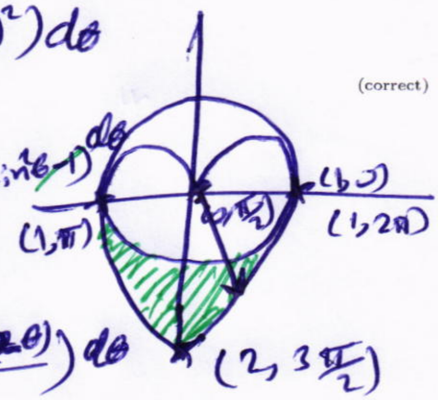
(a) $\frac{\pi}{4} + 2$ (correct)

(b) $\frac{\pi}{2} + 1$

(c) $\frac{\pi}{3} + 3$

(d) $\frac{\pi}{6} + 2$

(e) $\frac{\pi}{2} + 3$



Symmetry

$= \frac{1}{\cancel{2}} \cdot \cancel{2} \int_\pi^{3\pi/2} (\cancel{1} - 2\sin\theta + \sin^2\theta \cancel{-1})\, d\theta$

$= \int_\pi^{3\pi/2} \left(-2\sin\theta + \frac{1-\cos(2\theta)}{2}\right) d\theta$

$= 2\cos\theta \Big|_\pi^{3\pi/2} + \left(\frac{1}{2}\theta - \frac{\sin(2\theta)}{4}\right)\Big|_\pi^{3\pi/2}$

$= 2(0-(-1)) + \frac{1}{2}\left(\frac{3\pi}{2} - \pi\right) - (0-0)$

$= 2 + \frac{\pi}{4}.$

θ	$1-\sin\theta$
0	1
$\frac{\pi}{2}$	0
π	1
$\frac{3\pi}{2}$	2
2π	0

6. The area of the region enclosed by the inner loop of $r = 1 - \sqrt{2}\sin(\theta)$ is

Similar to Exercise 35 of 10.4.

(a) $\frac{\pi - 3}{2}$ (correct)

(b) $\frac{\pi + 1}{4}$

(c) $\frac{\pi + 5}{3}$

(d) $\frac{\pi + 7}{6}$

(e) $\frac{3\pi + 10}{6}$

$r = 0$, when $1 - \sqrt{2}\sin\theta = 0$.

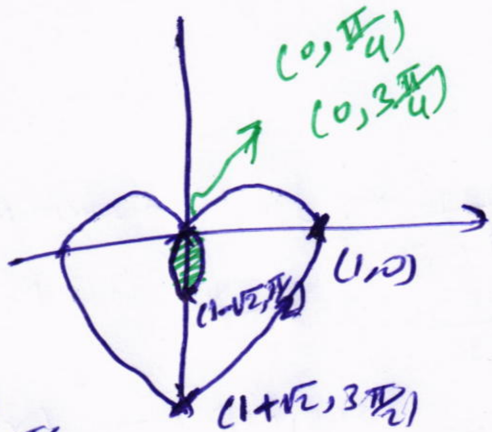
$\sin\theta = \frac{1}{\sqrt{2}}$

$\theta = \frac{\pi}{4} + 2k\pi$ or $\theta = \frac{3\pi}{4} + 2k\pi$ $\big(k \in \mathbb{Z}$

θ	$1-\sqrt{2}\sin\theta$
0	1
$\frac{\pi}{4}$	0
$\frac{\pi}{2}$	$1-\sqrt{2}$
$\frac{3\pi}{4}$	0
π	1
$\frac{3\pi}{2}$	$1+\sqrt{2}$

min r

max r



Using Symmetry

$A = \frac{1}{\cancel{2}} \cdot \cancel{2} \int_{\pi/4}^{\pi/2} (1 - \sqrt{2}\sin\theta)^2\, d\theta$

$= \int_{\pi/4}^{\pi/2} (1 - 2\sqrt{2}\sin\theta + 2\sin^2\theta)\, d\theta$

$= (\theta + 2\sqrt{2}\cos\theta)\Big|_{\pi/4}^{\pi/2} + \cancel{2} \cdot \int_{\pi/4}^{\pi/2} \frac{1-\cos(2\theta)}{\cancel{2}}\, d\theta$

$= \left(\frac{\pi}{2} - \frac{\pi}{4}\right) + 2\sqrt{2}\left(0 - \frac{1}{\sqrt{2}}\right) + \left(\theta - \frac{\sin(2\theta)}{2}\right)\Big|_{\pi/4}^{\pi/2}$

$= \cancel{\frac{\pi}{4}} - 2 + \left(\frac{\pi}{2} - \cancel{\frac{\pi}{4}}\right) - (0-1)/2 = \frac{\pi}{2} - \frac{3}{2}.$

7. The intersection of the sphere with center $(-1,4,2)$ and radius 7 with the xy-plane is

(a) the circle in the xy-plane with center $(-1,4,0)$ and radius $3\sqrt{5}$. (correct)
(b) the circle in the xy-plane with center $(-1,4,0)$ and radius $\sqrt{5}$.
(c) the circle in the xy-plane with center $(1,-4,0)$ and radius 3.
(d) the circle in the xy-plane with center $(0,4,2)$ and radius $3\sqrt{5}$.
(e) the circle in the xy-plane with center $(-1,0,2)$ and radius $3\sqrt{5}$.

Eqn. of the sphere $(x-(-1))^2+(y-4)^2+(z-2)^2=49$; xyplane: $z=0$

Setting $z=0$: $(x+1)^2+(y-4)^2+4=49$

$(x+1)^2+(y-4)^2=45=(3\sqrt{5})^2$.

8. The vector that has opposite direction to the vector $\vec{u} = <4,-3,0>$ but has length 10 is

(a) $<-8,6,0>$ (correct)
(b) $<8,6,0>$
(c) $<8,-6,0>$
(d) $<40,-30,0>$
(e) $<-40,30,0>$

$\vec{v} = -10 \cdot \dfrac{\vec{u}}{|\vec{u}|}$

$= -10 \cdot \dfrac{1}{\sqrt{16+9+0}} \cdot <4,-3,0>$

$= \dfrac{-10}{5} <4,-3,0>$

$= -2<4,-3,0>$

$= <-8,6,0>$.

9. The vector $\vec{\mathbf{v}}$ with length $\sqrt{8}$ that lies in the first quadrant and makes an angle $\dfrac{\pi}{4}$ with the positive x-axis is

$\vec{v} = \langle a, b \rangle$.

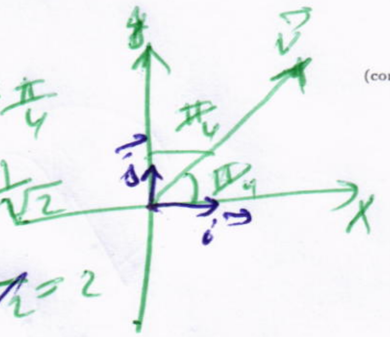
(a) $< 2, 2 >$ (correct)

(b) $< 2, 0 >$

(c) $\left\langle \dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}} \right\rangle$

(d) $\left\langle \dfrac{3}{\sqrt{2}}, \dfrac{1}{2} \right\rangle$

(e) $\left\langle \dfrac{1}{2}, \dfrac{3}{\sqrt{2}} \right\rangle$



$a = \vec{v} \cdot \vec{i} = |\vec{v}||\vec{i}| \cos\frac{\pi}{4}$

$= (\sqrt{8})(1) \cdot \frac{1}{\sqrt{2}}$

$= 2\sqrt{2} \cdot \frac{1}{\sqrt{2}} = 2$

$b = \vec{v} \cdot \vec{j} = |\vec{v}||\vec{j}| \cos\frac{\pi}{4}$

$= (\sqrt{8})(1) \frac{1}{\sqrt{2}}$

$= 2\sqrt{2} \cdot \frac{1}{\sqrt{2}}$

$= 2$

$\vec{v} = \langle a, b \rangle = \langle 2, 2 \rangle$

10. If the angle between the vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ is $\dfrac{\pi}{3}$, $|\vec{\mathbf{u}}| = 3$ and $|\vec{\mathbf{v}}| = 2$, then $|\vec{\mathbf{u}} + \vec{\mathbf{v}}| =$

(a) $\sqrt{19}$ (correct)

(b) $\sqrt{17}$

(c) $\sqrt{15}$

(d) $\sqrt{13}$

(e) 5

$|\vec{u} + \vec{v}|^2 = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v})$

$= \vec{u} \cdot \vec{u} + 2\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{v}$

$= |\vec{u}|^2 + 2|\vec{u}||\vec{v}| \cos\frac{\pi}{3} + |\vec{v}|^2$

$= 9 + (2)(3)\cancel{(2)} \cdot \frac{1}{\cancel{2}} + 4$

$= 19.$

$\therefore |\vec{u} + \vec{v}| = \sqrt{19}.$

11. Let $\vec{\mathbf{u}} < 1, -1, 2 >$, $\vec{\mathbf{v}} =< 3, -2, 0 >$. Then $(2\vec{\mathbf{u}} - \text{proj}_{\vec{\mathbf{v}}}\,\vec{\mathbf{u}}) \cdot \vec{\mathbf{v}} =$

(a) 5 (correct)

(b) $\frac{11}{6}$

(c) 6

(d) $\frac{6}{7}$

(e) $3\sqrt{2}$

$(\vec{u} - \text{proj}_{\vec{v}}\vec{u}) \perp \vec{v}$

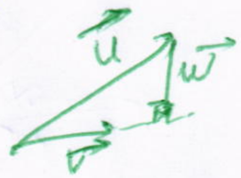
$\therefore (2\vec{u} - \text{proj}_{\vec{v}}\vec{u}) \cdot \vec{v}$

$= (\vec{u} + (\vec{u} - \text{proj}_{\vec{v}}\vec{u})) \cdot \vec{v}$

$= \vec{u} \cdot \vec{v} + 0$

$= (1)(3) + (-1)(-2) + (2)(0)$

$= 5$



$\vec{u} = \text{proj}_{\vec{v}}\vec{u} + \vec{v}$

$\vec{w} = (\vec{u} - \text{proj}_{\vec{v}}\vec{u}) \cdot \vec{v} = 0.$

Exercise 45 of 12.3

12. Consider the points $A(2, 1, -1)$, $B(3, 0, 2)$, $C(4, -2, -1)$, $D(3, m, 0)$. The value of m that makes all four points coplanar is

Similar to Example 5 of 12.4

(a) $\frac{-1}{3}$ (correct)

(b) $\frac{1}{3}$

(c) $\frac{-1}{2}$

(d) $\frac{1}{2}$

(e) 1

The points are coplanar iff the volume of the parallelpiped formed by them is zero.

We find $\vec{AB} = \langle 1, -1, 3 \rangle$, $\vec{AC} = \langle 2, -3, 0 \rangle$

$\vec{AD} = \langle 1, m-1, 1 \rangle$.

$V = \vec{AB} \cdot (\vec{AC} \times \vec{AD})$

$0 = \begin{vmatrix} 1 & -1 & 3 \\ 2 & -3 & 0 \\ 1 & m-1 & 1 \end{vmatrix} = 1(-3-0) - (-1)(2-0) + 3(2(m-1) - (-3))$

$\therefore 0 = -3 + 2 + 6m + 3$

$-2 = 6m$

$\boxed{m = -\frac{1}{3}}$

13. The set of all points at which the parametric curve

$$x = t^3 - 3t,\ \ y = t^2 - 3,\ t \in (-\infty, \infty)$$

has a vertical tangent is

$\frac{dy}{dt} = 2t$; $\frac{dx}{dt} = 3t^2 - 3$

(a) $\{(2,-2), (-2,-2)\}$. (correct)
(b) $\{(2,-2)\}$.
(c) $\{(-2,-2)\}$.
(d) $\{(0,-3)\}$.
(e) $\{(0,3)\}$.

We have a vertical tangent when $\frac{dx}{dt} = 0$ & $\frac{dy}{dt} \neq 0$.

$\frac{dx}{dt} = 0 \Leftrightarrow 3t^2 - 3 = 0 \Leftrightarrow t^2 = 1 \Leftrightarrow t = \mp 1$

must be checked

$\boxed{\text{At these values of } t;\ \frac{dy}{dt} = \mp 2 \neq 0}$.

So, the curve has a vertical tangent at $t = \mp 1$ equivalently $(x, y) = (\mp 2, -2)$.

14. The set of all values of b that makes the angle between the vectors $< 2, 1, -1 >$ and $< 1, b, 0 >$ equal to $\frac{\pi}{4}$ is

(a) $\left\{\frac{2+\sqrt{6}}{2}, \frac{2-\sqrt{6}}{2}\right\}$ (correct)
(b) $\left\{\frac{2+\sqrt{6}}{2}\right\}$
(c) $\left\{\frac{2-\sqrt{6}}{2}\right\}$
(d) $\left\{\frac{4+\sqrt{6}}{2}\right\}$
(e) $\left\{\frac{4-\sqrt{6}}{2}\right\}$

$\vec{u} \cdot \vec{v} = |\vec{u}|\,|\vec{v}| \cos\theta$

$2 + b - 0 = \sqrt{6} \cdot \sqrt{1+b^2} \cdot \frac{1}{\sqrt{2}}$

$(2+b)^2 = \frac{(6)(1+b^2)}{2} = 3(1+b^2)$

$4 + 4b + b^2 = 3 + 3b^2$

$2b^2 - 4b - 1 = 0.$

$b = \frac{-(-4) \mp \sqrt{(-4)^2 - 4(2)(-1)}}{2(2)}$

$= \frac{4 \mp \sqrt{16+8}}{4} = \frac{2 \mp \sqrt{6}}{2}$

15. The distance from the origin to the line passing through the points $(0, 1, 3)$ Q and $(1, -1, 2)$ R is

Similar to Exercise 45 of 12.4

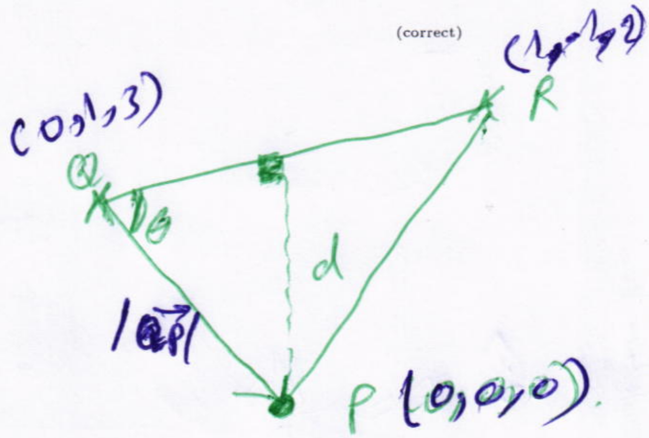
(a) $\sqrt{\dfrac{35}{6}}$ (correct)

(b) $\dfrac{\sqrt{11}}{3}$

(c) $\dfrac{\sqrt{22}}{3}$

(d) 3

(e) 2



Area of $\triangle QRP = \frac{1}{2}|\vec{QR}|\, d$

$\cancel{\tfrac{1}{2}}\,|\vec{QR}\times\vec{QP}| = \cancel{\tfrac{1}{2}}\,|\vec{QR}|\,d$

$\therefore\ d = \dfrac{|\vec{QR}\times\vec{QP}|}{|\vec{QR}|}$

$\vec{QR} = \langle 1, -2, -1\rangle$

$\vec{QP} = \langle 0, -1, -3\rangle$

$\vec{QR}\times\vec{QP} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & -1 \\ 0 & -1 & -3 \end{vmatrix}$

$= \langle 5, 3, -1\rangle$

$\therefore\ d = \dfrac{\sqrt{25+9+1}}{\sqrt{1+4+1}}$

$= \dfrac{\sqrt{35}}{\sqrt{6}}$

$= \sqrt{\dfrac{35}{6}}$

OR

$\sin\theta = \dfrac{d}{|\vec{QP}|}$

$d = |\vec{QP}|\sin\theta$

$= \dfrac{|\vec{QR}|\,|\vec{QP}|\sin\theta}{|\vec{QR}|}$

$= \dfrac{|\vec{QR}\times\vec{QP}|}{|\vec{QR}|}$

Formula given in Exercise 45 of section 12.4

$= \dfrac{\sqrt{25+9+1}}{\sqrt{1+4+1}}$

$= \sqrt{\dfrac{35}{6}}$