Similar to Example 5 in 12-5

1. If $3x + by + cz = d$ is an equation of the plane containing the points $A(2,0,1)$, $B(4,2,7)$ and $C(-1,-1,1)$, then $b + c + d =$

(a) 1 (correct)

(b) 3

(c) -3

(d) -1

(e) 4

$$\vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & 6 \\ -3 & -1 & 0 \end{vmatrix} = \langle 6, -18, 4 \rangle$$

An eqn of the plane using $\vec{n}$ & A:

$$6(x-2) - 18(y-0) + 4(z-1) = 0.$$

$$\boxed{3x - 9y + 2z = 8} \quad \therefore \; b + c + d = -9 + 2 + 8 = 1.$$

Similar to Example 3 in 12.5

2. The lines

$L_1 := x = 3 + 2t,\ y = 4 - t,\ z = 1 + 3t,\ t \in (-\infty, \infty)$

$L_2 := x = 1 + 4s,\ y = 3 - 2s,\ z = 4 + 5s,\ s \in (-\infty, \infty)$

are

choose $\vec{v_1} = \langle 2, -1, 3 \rangle \parallel L_1$

$\vec{v_2} = \langle 4, -2, 5 \rangle \parallel L_2$.

(a) skew (correct)

(b) parallel

(c) intersect and are perpendicular

(d) intersect at the point $(5, 3, 4)$

(e) intersect at the point $(5, 1, 9)$

$\vec{v_1} \nparallel \vec{v_2}$, since $\vec{v_1} \times \vec{v_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 3 \\ 4 & -2 & 5 \end{vmatrix} = \langle 1, 2, 0 \rangle \neq \vec{0}$

Suppose (x_0, y_0, z_0) is a point of intersection.

$\therefore$ $x_0 = 3 + 2t = 1 + 4s$, whence $t = 2s - 1$ — ①

$y_0 = 4 - t = 3 - 2s$, whence $t = 2s + 1$ — ②

From ① & ②: $2s - 1 = 2s + 1$, so $-1 = 1$ a contradiction.

$\therefore$ $L_1 \nparallel L_2$ & $L_1 \cap L_2 = \{\} = \emptyset$, whence L_1 & L_2 are skew.

Similar to Example 7 in 12.6

3. The set of **all** values of k such that

$$-x^2 + 2x - y^2 - 2y + z^2 = k + 1$$

represents a hyperboloid of two sheets is

(a) $(1, \infty)$ (correct)
(b) $\{2\}$
(c) $\{1\}$
(d) $(-\infty, 1)$
(e) $\{-1\}$

Completing the square, we obtain

$$-(x^2 - 2x + 1) - (y^2 + 2y + 1) + z^2 = k + 1 - 2$$

$$z^2 - (x-1)^2 - (y+1)^2 = k - 1.$$

To have a hyperboloid of two sheets we must have $k - 1 > 0$, i.e. $k > 1$.

Exercise 67 in 14.2

4. The level surfaces of

$$f(x, y, z) = x + 3y + 5z$$

are

(a) a family of parallel planes with a normal vecter $< 1, 3, 5 >$. (correct)
(b) a family of cocentric spheres with center at the origin.
(c) a family of hyperboloids of one sheet.
(d) a family of hyperboloids of two sheets.
(e) a family of right circular cones.

the range of the function is $\mathbb{R}$

$$x + 3y + 5z = k, \quad k \in \mathbb{R}$$

represents a family of planes each of which is parallel to the others since all have the normal vector

$$\vec{n} = \langle 1, 3, 5 \rangle.$$

Similar to Ex 20 in 14.1

5. The domain of

$$f(x,y) = \sin^{-1}(x^2 + y^2 - 2)$$

is

(a) $\{(x,y) |\ 1 \le x^2 + y^2 \le 3\}$ (correct)
(b) $\{(x,y) |\ 2 \le x^2 + y^2 \le 3\}$
(c) $\{(x,y) |\ x > 0,\ y > 0\}$
(d) $\{(x,y) |\ y > 0\}$
(e) $\{(x,y) |\ x > 0\}$

The domain of $g(u) = \sin^{-1}(u)$ is $[-1,1]$

So, Domain (f) $= \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid -1 \le x^2 + y^2 - 2 \le 1\}$

$= \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid +1 \le x^2 + y^2 \le 3\}$

Exercise 15 in 14.3

6. $$\lim_{(x,y)\to(0,0)} \frac{xy^2 \cos y}{x^2 + y^4}$$

(a) Does Not Exist (correct)
(b) equals 1
(c) equals $\frac{1}{2}$
(d) equals 0
(e) equals ∞

- Along the path $x = 0$:

$$L = \lim_{y\to 0} \frac{(0)\underbrace{(y^2 \cos y)}_{\text{bounded near } 0}}{0 + y^4} = \lim_{y\to 0} \frac{0}{y^4} = 0.$$

or

- Along the path $y = 0$:

$$L = \lim_{x\to 0} \frac{\overbrace{x}^{\text{bounded near } 0}(0)(\cos(0))}{x^2 + 0} = \lim_{x\to 0} \frac{0}{x^2} = 0.$$

- However, along the path $x = y^2$, we have

$$L = \lim_{y\to 0} \frac{y^2 \cdot y^2 \cdot \cos y}{(y^2)^2 + y^4} = \lim_{y\to 0} \frac{y^4 \cos y}{2y^4} = \frac{1}{2} \ne 0.$$

Since the limit is path dependent, it DNE.

Similar to Ex. 29 in 14.3

7. One of the points at which the function

$$f(x,y) = \frac{xy}{1 - e^{x+y}}$$

is **not** continuous is

(a) $(1,-1)$ (correct)
(b) $(1,1)$
(c) $(-1,-1)$
(d) $(-1,0)$
(e) $(0,-1)$

The function is cts. on $\mathbb{R}\times\mathbb{R} \setminus \{(x,y) \in \mathbb{R}\times\mathbb{R} \mid x+y=0\}$

Notice that $(1,-1)$ is the only given choice satisfying $x+y=0$

Similar to Example 9 in 14.3

8. If $u(x,y) = e^{2x}\sin(2y)$, then $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} =$

(a) 0 (correct)
(b) 2
(c) 4
(d) 8
(e) -8

$\frac{\partial u}{\partial x} = 2e^{2x}\sin(2y)$

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(2e^{2x}\sin(2y)\right)$

$\boxed{\frac{\partial^2 u}{\partial x^2} = 4e^{2x}\sin(2y)}$

$\frac{\partial u}{\partial y} = 2e^{2x}\cos(2y)$

$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(2e^{2x}\cos(2y)\right)$

$\boxed{\frac{\partial^2 u}{\partial y^2} = -4e^{2x}\sin(2y)}$

So, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{2x}\sin(2y)(4-4)$
$= 0$

Exercise 44 in 14.3

9. If $f(x,y,z) = x^{yz}$, then $f_z(e,1,0) =$

(a) 1
(b) 0
(c) e
(d) e^{-1}
(e) -1

$w = x^{yz}$, whence $\ln(w) = yz \ln(x)$ (correct)

$(x,y,z \geq 0)$ $\ln(w) = yz\ln(x)$.

$\frac{1}{w}\frac{\partial w}{\partial z} = y\ln(x)$.

$$\boxed{\frac{\partial w}{\partial z} = w\,y\ln(x) = x^{yz}\cdot y\ln(x)}$$

$f_z(e,1,0) = e^{(1)(0)}\cdot 1\cdot \ln(e) = 1$.

Similar to Exercises 1–6 in 14.4

10. The tangent plane to the surface $2x + y^2 - z^2 = 0$ at the point $(0,1,1)$ contains the point

(a) $(-1,1,0)$
(b) $(0,1,-1)$
(c) $(1,0,2)$
(d) $\left(\frac{1}{2},\frac{1}{2},0\right)$
(e) $\left(-\frac{1}{2},-\frac{1}{2},0\right)$

$f(x,y,z) = 2x + y^2 - z^2$. (correct)

$\vec{\nabla} f = \langle 2, 2y, -2z\rangle$.

$\vec{\nabla} f\big|_{(0,1,1)} = \langle 2, 2, -2\rangle$.

The equation of the tangent plane is:

$$2(x-0) + 2(y-1) - 2(z-1) = 0.$$

$$2x + 2y - 2z = 0.$$

$$\boxed{x + y - z = 0}.$$

Notice that the only given choice satisfying the eqn. is $(-1,1,0)$.

Similar to Ex 20 & 21 in 14.4

11. If $L(x,y)$ is the linearization of $f(x,y) = \dfrac{y-1}{x+1}$ at $(0,0)$, then $L(0.1,-0.2) =$

(a) -1.1 (correct)

(b) -1

(c) -0.9

(d) 1

(e) 1.1

$f_x = \dfrac{-(y-1)}{(x+1)^2}$, $f_y = \dfrac{1}{x+1}$.

$f_x\big|_{(0,0)} = 1$, $f_y\big|_{(0,0)} = 1$.

$L(x,y) = f(0,0) + f_x\big|_{(0,0)}(x-0) + f_y\big|_{(0,0)}(y-0)$

$\boxed{L(x,y) = -1 + x + y}$

$L(0.1,-0.2) = -1 + 0.1 - 0.2 = -1.1$

Similar to Example 5 in 14.5

12. Let

$$u = x^2\sin(y) + ye^{xy},\ x = s + 2t \text{ and } y = st.$$

Then the value of $\dfrac{\partial u}{\partial s}$ at $(s,t) = (0,1)$ is

(a) 5 (correct)

(b) 6

(c) 0

(d) 4

(e) 8

$\dfrac{\partial u}{\partial s} = \dfrac{\partial u}{\partial x}\cdot\dfrac{\partial x}{\partial s} + \dfrac{\partial u}{\partial y}\cdot\dfrac{\partial y}{\partial s}$

$= (2x\sin(y) + y^2e^{xy})\cdot 1$

$+ (x^2\cos(y) + (e^{xy} + y^2e^{xy}))\cdot t$

$\dfrac{\partial u}{\partial s}\Big|_{\substack{(s,t)=(0,1)\\(x,y)=(2,0)}} = (0+0)\cdot 1 + (4+1+0)\cdot 1$

$= 0 + 5$

$= 5.$

Similar to Exercise 15 in 14-5

13. Suppose that f is a differentiable function of x and y and that

$$g(u,v) = f(e^{3u} + \sin(2v),\, e^{4u} + \cos(v)).$$

If $f_x(1,2) = 2$ and $f_y(1,2) = 5$, then $g_u(0,0) =$

(a) 26 (correct)

(b) 7

(c) 6

(d) 20

(e) 36

Consider $x = e^{3u} + \sin(2v)$, $y = e^{4u} + \cos(v)$

$g_u = f_x \cdot \frac{\partial x}{\partial u} + f_y \cdot \frac{\partial y}{\partial u}$

$= f_x \cdot 3e^{3u} + f_y \cdot 4e^{4u}$

$g_u(0,0) = (f_x(1,2))(3e^0) + f_y(1,2) \cdot (4e^0)$

$= (2)(3) + (5)(4) = 26$

$(u,v) = (0,0)$

$\Rightarrow (x,y) = (1,2)$

Similar to Exercises 19 & 20 in 14.6

14. The directional derivative of $f(x,y,z) = \sqrt{yz^2 + xy}$ at $P(1,2,-1)$, in the direction of $Q(-1,3,1)$, is

(a) $-\frac{5}{6}$ (correct)

(b) $-\frac{1}{3}$

(c) $-\frac{2}{3}$

(d) $-\frac{1}{2}$

(e) $-\frac{1}{6}$

$\vec{\nabla} f = \langle f_x, f_y, f_z \rangle.$

$= \left\langle \frac{y}{2\sqrt{yz^2+xy}}, \frac{z^2+x}{2\sqrt{yz^2+xy}}, \frac{2yz}{2\sqrt{yz^2+xy}} \right\rangle$

$\vec{\nabla} f \Big|_{P(1,2,-1)} = \left\langle \frac{2}{2(2)}, \frac{1+1}{2(2)}, \frac{-4}{2(2)} \right\rangle$

$= \left\langle \frac{1}{2}, \frac{1}{2}, -1 \right\rangle.$

$\vec{u} = \frac{\vec{PQ}}{|\vec{PQ}|} = \frac{\langle -2, 1, 2 \rangle}{\sqrt{9}} = \left\langle -\frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right\rangle$

$D_{P,\vec{u}} f = \vec{\nabla} f\Big|_P \cdot \vec{u} = \left(\frac{1}{2}\right)\left(-\frac{2}{3}\right) + \frac{1}{2}\left(\frac{1}{3}\right) + (-1)\left(\frac{2}{3}\right)$

$= -\frac{1}{3} + \frac{1}{6} - \frac{2}{3} = -\frac{5}{6}.$

Similar to Exercise 33 in 14.6

15. Suppose that over a certain region of space, the electrical potential V is given by $V = 2y^2 - yz + x^2yz$. If the vector $\vec{u}$ is the direction in which V changes most rapidly at $P(1,1,2)$, and M is the maximum rate of change of V at P, then

(a) $\vec{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle$ and $M = 4\sqrt{2}$ (correct)

(b) $\vec{u} = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$ and $M = 4\sqrt{3}$

(c) $\vec{u} = \left\langle \frac{4}{\sqrt{21}}, \frac{2}{\sqrt{21}}, \frac{1}{\sqrt{21}} \right\rangle$ and $M = \sqrt{21}$

(d) $\vec{u} = \left\langle \frac{3}{\sqrt{17}}, \frac{2}{\sqrt{17}}, \frac{-2}{\sqrt{17}} \right\rangle$ and $M = \sqrt{17}$

(e) $\vec{u} = \left\langle \frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}} \right\rangle$ and $M = \sqrt{14}$

- $\vec{\nabla} V = \langle V_x, V_y, V_z \rangle$

$= \langle 2xyz, 4y - z + x^2 z, -y + x^2 y \rangle$

$\vec{\nabla} V \big|_{P(1,1,2)} = \langle 4, 4, 0 \rangle.$

- The maximum rate of change of V at P is $|\vec{\nabla} V_P| = \sqrt{4^2 + 4^2 + 0^2} = \sqrt{32} = 4\sqrt{2}$

in the direction of $\vec{\nabla} V$, which is $\vec{u} = \frac{\vec{\nabla} V}{|\vec{\nabla} V|} = \frac{4\langle 1,1,0\rangle}{4\sqrt{2}}$

$= \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \rangle$