1. The slope of the tangent line to the polar curve $r = 2\cos(\theta)$ at $\theta = \frac{\pi}{3}$ is

(a) $\frac{1}{\sqrt{3}}$ (correct)
(b) $\sqrt{3}$
(c) $\frac{1}{\sqrt{2}}$
(d) $\sqrt{2}$
(e) 1

$$m_{\text{tangent}} = \frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

$$\frac{dr}{d\theta} = -2\sin\theta$$

$$\left.\frac{dr}{d\theta}\right|_{\theta=\frac{\pi}{2}} = -2\cdot\frac{\sqrt{3}}{2} = -\sqrt{3}.$$

$$r(\tfrac{\pi}{3}) = 2\cdot\tfrac{1}{2} = 1$$

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = \frac{-\sqrt{3}\cdot\frac{\sqrt{3}}{2} + (1)\frac{1}{2}}{-\sqrt{3}\cdot\frac{1}{2} - (1)\frac{\sqrt{3}}{2}} = \frac{-1}{-\sqrt{3}} = \frac{1}{\sqrt{3}}$$

2. If (a, b, c) is the point of intersection between the plane $2x + y + z = 9$ and the line through $(1, 0, 1)$ and $(2, -1, 3)$, then $a + b + c =$

(a) 6 (correct)
(b) -4
(c) 0
(d) -2
(e) 10

$L \parallel \vec{v} = \langle 1, -1, 2 \rangle$

$L: x = 1 + t$
$y = 0 - t$
$z = 1 + 2t$, $t \in \mathbb{R}$

Setting the values of x, y & z in the equation of the plane yields:

$$2x + y + z = 9$$
$$2(1+t) + (-t) + (1+2t) = 9$$
$$3t = 6; \quad t = 2.$$

$\therefore (a, b, c) = (1+2, -2, 1+2(2))$
$= (3, -2, 5)$. So, $a+b+c = 6$.

3. If $T = \frac{v}{2u+v}$, $u = pq\sqrt{r}$ and $v = p\sqrt{q}r$, then the value of $\frac{\partial T}{\partial q}$ when $(p,q,r) = (2,1,4)$ is

A $(u,v) = (4,8)$.

(a) $-\frac{1}{8}$ (correct)

(b) $\frac{1}{8}$

(c) $\frac{1}{4}$

(d) $-\frac{1}{4}$

(e) 0

$$\frac{\partial T}{\partial q} = \frac{\partial T}{\partial u}\cdot\frac{\partial u}{\partial q} + \frac{\partial T}{\partial v}\cdot\frac{\partial v}{\partial q}$$

$$= \frac{0-2(v)}{(2u+v)^2}\cdot p\sqrt{r} + \frac{1(2u+\cancel{v})-1(\cancel{v})}{(2u+v)^2}\cdot\frac{pr}{2\sqrt{q}}$$

$$\left.\frac{\partial T}{\partial q}\right|_A = \frac{-\cancel{2}(8)}{(\cancel{16})(16)}\cdot(2)(2) + \frac{(2)(4)}{(16)(16)}\cdot\frac{\cancel{(2)}(4)}{\cancel{(2)}(1)}$$

$$= \frac{-1}{4} + \frac{1}{8} = \frac{-1}{8}$$

4. The number of points at which the **normal line** through the point $(1,-1,1)$ on the ellipsoid $x^2+2y^2+2z^2=5$ intersects the sphere $x^2+y^2+(z-1)^2=5$ is

(a) 2 (correct)

(b) 1

(c) 0

(d) 3

(e) 4

Consider $f(x,y,z) = x^2+2y^2+2z^2$.

normal line $\parallel \nabla f|_{(1,-1,1)} = \langle 2x, 4y, 4z\rangle|_{(1,-1,1)}$

$= \langle 2,-4,4\rangle$.

Equations of the normal line:

$P_1\ (0,1,3),\ t=-\frac{1}{2}$

$P_2\ (\frac{4}{3}, -\frac{5}{3}, \frac{5}{3}),\ t=\frac{1}{6}$

$$\left.\begin{aligned} x &= 1+2t \\ y &= -1-4t = -(1+4t) \\ z &= 1+4t. \end{aligned}\right\}\ t\in\mathbb{R}$$

Substituting in the equation of the sphere:

$t=-\frac{1}{2}$ or $t=\frac{1}{6}$

$x^2+y^2+(z-1)^2=5$; $(1+2t)^2+(1+4t)^2+(4t)^2=5$

$36t^2+12t-3=0$; $12t^2+4t-1=0$.

$t = \frac{-4\mp\sqrt{16-(4)(12)(-1)}}{24} = \frac{-4\mp\sqrt{64}}{24}$

5. If $(1,1)$ is a critical point of $f(x,y) = x^4 + y^4 - 4xy + 1$, then

(a) f has a local minimum at $(1,1)$ (correct)
(b) f has a local maximum at $(1,1)$
(c) f has a saddle point at $(1,1)$
(d) f has an absolute maximum at $(1,1)$
(e) none of the above

$f_x = 4x^3 - 4y$

$f_y = 4y^3 - 4x$

$f_x(1,1) = 0 = f_y(1,1)$

$f_{xx} = 12x^2;\ f_{yy} = 12y^2;\ f_{xy} = -4 = f_{yx}$

$D(x,y) = f_{xx} \cdot f_{yy} - (f_{xy})^2 = 144x^2y^2 - 16.$

$D(1,1) = 144 - 16 > 0.$ Since $f_{xx}(1,1) = 12 > 0$, $f(1,1)$ is local minimum.

6. The absolute minimum m and the absolute maximum M of $f(x,y) = x^2 + y^2 + x^2y + 4$ on $D = \{(x,y) \mid |x| \le 1,\ |y| \le 1\}$ are

(a) $m = 4,\ M = 7$ (correct)
(b) $m = 5,\ M = 7$
(c) $m = 4,\ M = 9$
(d) $m = 4,\ M = 6$
(e) $m = 5,\ M = 9$

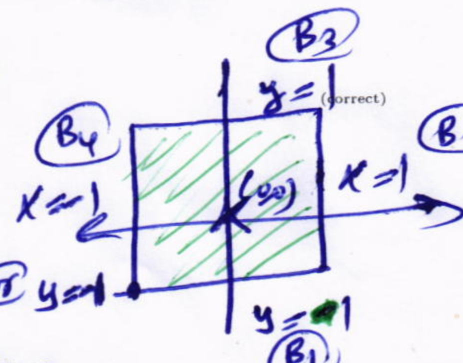
(x,y)	$f(x,y)$
$(0,0)$	4
$(0,1)$	5
$(1,-\frac{1}{2})$	19/4
$(-1,-\frac{1}{2})$	19/4
$(-1,-1)$	5
$(1,1)$	7
$(-1,1)$	7
$(1,-1)$	5

→ absolute min.

→ absolute max

$f_x = 2x(1+y)$

$f_y = 2y + x^2.$



$f_x = 0 \Leftrightarrow x = 0$ or $y = -1$

$x = 0 : f_y = 0 \Leftrightarrow y = 0$

$y = -1 : f_y = 0 \Leftrightarrow x = \mp\sqrt{2}$ (rejected)

$\therefore f$ has only one critical point $(0,0)$.

$B_1 : \ y = -1$ $\quad f(x,-1) = f_1(x) = 5$ constant function

$B_2 : \ y = 1$ $\quad f(x,1) = f_2(x) = 2x^2 + 5;\ x \in [-1,1]$

$f_2'(x) = 2x;\ f_2'(x) = 0 \Leftrightarrow x = 0.$ $(0,1)$

$B_3 : \ x = \mp 1$
B_4

$f_3(y) = f(\mp 1, y) = y^2 + y + 5;\ y \in [-1,1]$

$f_3'(y) = 2y + 1;\ f_3'(y) = 0 \Leftrightarrow y = -\frac{1}{2}.$

$(1,-\frac{1}{2})$ $(-1,-\frac{1}{2})$

7. If m is the absolute minimum and M is the absolute maximum of $f(x,y,z) = y^2 - 10z$ subject to $\underbrace{x^2+y^2+z^2}_{g(x,y,z)} = 36$, then $m + M =$ $-60+61 = 1.$

(a) 1 (correct)

(b) 2

(c) 0

(d) −2

(e) −1

$\nabla f = \lambda \nabla g$

$\langle 0, 2y, -10\rangle = \lambda \langle 2x, 2y, 2z\rangle.$

$0 = 2\lambda x$ — ① ~~$\lambda = 0$~~ rejected or $x = 0$.

$2y = 2\lambda y$ — ② $2y(1-\lambda)=0$

$\boxed{-10 = 2\lambda z}$ — ③ $\lambda \neq 0$. $\boxed{\text{So, } x = 0}$

From ② $y = 0$ or $\lambda = 1$. If $y = 0$, then $z = \mp 6$.

If $\lambda = 1$; $z = -5$ from 3. $y^2 = 11$ $f(0,y,-5) = 11+50 = 61$ $f(0,0,6) = -60$. $f(0,0,-6) = 60$.

8. The average value of $f(x,y) = 4x + 6y + 15$ over $R = [-1,2] \times [-1,1]$ is

(a) 17 (correct)

(b) 15

(c) 13

(d) 11

(e) 9

$$\int_{-1}^{1}\int_{-1}^{2} (4x+6y+15)\,dx\,dy$$

$$= \int_{-1}^{1} \left(2x^2 + 6yx + 15x\right)\Big|_{-1}^{2} dy$$

$$= \int_{-1}^{1} \left((8+12y) - (-13-6y)\right) dy$$

$$= \int_{-1}^{1} (51 + 18y)\,dy$$

$$= \left(51y + 9y^2\right)\Big|_{-1}^{1} = 102.$$

$$\text{Average}(f)\Big|_R = \frac{\iint_R f(x,y)\,dA}{\text{Area}(A)} = \frac{102}{(3)(2)} = 17$$

9. An estimation of $\iint_R xe^{-xy}dA$ where the rectangle $R = [0,2]\times[0,1]$ is divided into four equal rectangles and the sample points are chosen to be the **upper right corners**, is

(Announced Clarification: Divide each of the horizontal and vertical intervals into two subintervals)

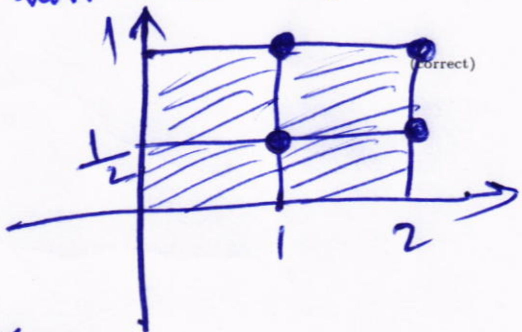
(a) $\frac{1}{2}e^{\frac{-1}{2}} + \frac{3}{2}e^{-1} + e^{-2}$ (correct)

(b) $\frac{1}{2}e^{\frac{-1}{2}} + e^{-1} + e^{-2}$

(c) $\frac{1}{2}e^{\frac{-1}{2}} + 2e^{-1} + 2e^{-2}$

(d) $\frac{1}{2}e^{\frac{-1}{2}} + \frac{1}{2}e^{-1} + \frac{1}{2}e^{-2}$

(e) $\frac{1}{2}e^{\frac{-1}{2}} + \frac{3}{2}e^{-1} + \frac{1}{2}e^{-2}$



$$\iint_R xe^{-xy} \approx \frac{1}{2}\left(f(1,\tfrac{1}{2}) + f(2,\tfrac{1}{2}) + f(1,1) + f(2,1)\right)$$

$$= \frac{1}{2}\left(e^{-\frac{1}{2}} + 2e^{-1} + e^{-1} + 2e^{-2}\right)$$

$$= \frac{1}{2}e^{-\frac{1}{2}} + \frac{3}{2}e^{-1} + e^{-2}.$$

10. $\displaystyle\int_{-1}^{1}\int_{0}^{1} \frac{xy^6}{1+x^4}\,dy\,dx =$

(a) 0 (correct)

(b) 1

(c) -1

(d) 2

(e) -2

$$\int_{-1}^{1}\left(\frac{x}{1+x^4}\left(\frac{y^7}{7}\Big|_0^1\right)\right)dx$$

$$= \frac{1}{7}\int_{-1}^{1}\frac{x}{1+x^4}\,dx = 0.$$

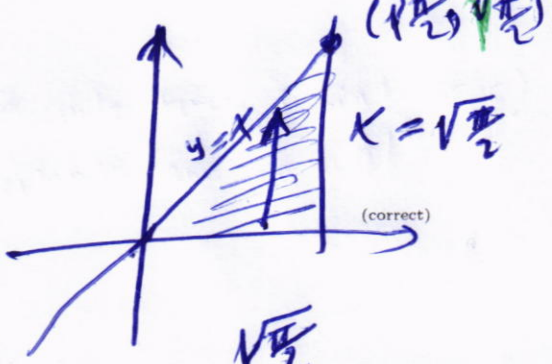
odd function

or

$$I = \Big(\underbrace{\int_{-1}^{1}\frac{x}{1+x^4}\,dx}_{\text{odd}}\Big)\left(\int_0^1 y^6\,dy\right)$$

$$= (0)\int_0^1 y^6\,dy = 0.$$

11. $\displaystyle\int_0^{\sqrt{\frac{\pi}{2}}}\int_y^{\sqrt{\frac{\pi}{2}}} \cos(x^2)\,dx\,dy =$



$\displaystyle\int_0^{\sqrt{\frac{\pi}{2}}}\int_0^{x} \cos(x^2)\,dy\,dx$

(a) $\frac{1}{2}$ (correct)

(b) 1

(c) 0

(d) $-\frac{1}{2}$

(e) -1

$\displaystyle = \int_0^{\sqrt{\frac{\pi}{2}}} \left(\cos(x^2)\,y\Big|_0^x\right) dx = \int_0^{\sqrt{\frac{\pi}{2}}} x\cos(x^2)\,dx$

$\displaystyle = \frac{\sin(x^2)}{2}\Big|_0^{\sqrt{\frac{\pi}{2}}}$

$= \frac{1}{2}.$

12. If S is the region bounded by the y-axis and the right half of the unit circle, then the value of

$$I = \iint_S \sqrt{4 - x^2y^2}\,dA$$

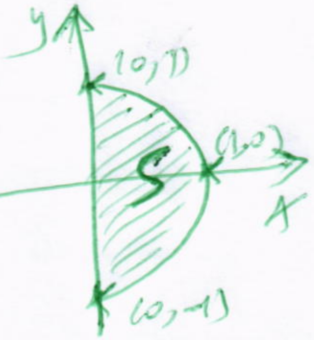
lies in the interval

clearly $0 \le x^2y^2 \le 1$

$-1 \le -x^2y^2 \le 0$

$3 \le 4 - x^2y^2 \le 4$

$\sqrt{3} \le \sqrt{4-x^2y^2} \le 2$



(a) $\left[\frac{\sqrt{3}}{2}\pi,\, \pi\right]$ (correct)

(b) $\left(\pi,\, \frac{\sqrt{5}}{2}\pi\right)$

(c) $\left[\frac{\sqrt{5}}{2}\pi,\, 2\pi\right]$

(d) $\left[0, \frac{\pi}{2}\right]$

(e) $\left(\frac{\pi}{2}, \frac{\sqrt{3}}{2}\pi\right)$

$\displaystyle \therefore \iint_S \sqrt{3}\,dA \le I \le \iint_S 2\,dA$

$\sqrt{3}\cdot\frac{\pi}{2} \le I \le 2\cdot\frac{\pi}{2}$

Area of the half circle $= \frac{\pi(1)^2}{2} = \frac{\pi}{2}$

$\frac{\sqrt{3}}{2}\pi \le I \le \pi$

13. $\displaystyle\int_0^{\frac{1}{2}}\int_{\sqrt{3}y}^{\sqrt{1-y^2}} xy^2\,dx\,dy =$

$$\int_0^{1/2}\left(\frac{x^2}{2}\Big|_{\sqrt{3}y}^{\sqrt{1-y^2}} y^2\right)dy$$

(a) $\frac{1}{120}$

(b) $\frac{1}{100}$

(c) $\frac{1}{80}$

(d) $\frac{1}{60}$

(e) $\frac{1}{40}$

(correct)

$$= \int_0^{1/2} \frac{y^2}{2}\cdot(1-y^2-3y^2)\,dy$$

$$= \frac{1}{2}\int_0^{1/2}(y^2-4y^4)\,dy$$

$$= \frac{1}{2}\left(\frac{y^3}{3}-\frac{4y^5}{5}\right)\Big|_0^{1/2}$$

$$= \frac{1}{2}\left(\frac{1}{24}-\frac{1}{40}\right) = \frac{1}{2}\left(\frac{5-3}{120}\right) = \frac{1}{120}$$

14. If R is the region in the **first quadrant** between the circles $x^2+y^2=\frac{\pi}{3}$ and $x^2+y^2=\frac{\pi}{2}$, then $\displaystyle\iint\limits_R \sin(x^2+y^2)\,dA =$

$$\int_0^{\pi/2}\int_{\sqrt{\pi/3}}^{\sqrt{\pi/2}} \sin(r^2)\,\underbrace{\boxed{r}}_{\text{Jacobian}}\,dr\,d\theta$$

(a) $\frac{\pi}{8}$

(b) $\frac{\pi}{6}$

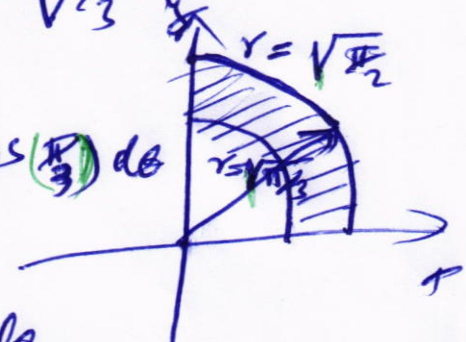
(c) $\frac{\pi}{4}$

(d) $\frac{\pi}{2}$

(e) π

(correct)

$$= -\int_0^{\pi/2}\left(\frac{\cos(r^2)}{2}\Big|_{\sqrt{\pi/3}}^{\sqrt{\pi/2}}\right)d\theta$$



$$= -\frac{1}{2}\int_0^{\pi/2}\left(\cos\left(\tfrac{\pi}{2}\right)-\cos\left(\tfrac{\pi}{3}\right)\right)d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2} -\frac{1}{2}\,d\theta$$

$$= \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)\int_0^{\pi/2} d\theta = \frac{1}{4}\left(\theta\Big|_0^{\pi/2}\right)$$

$$= \left(\frac{1}{4}\right)\left(\frac{\pi}{2}\right)$$

$$= \frac{\pi}{8}.$$

15. The volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$ and $z = 0$ is

(a) $\frac{1}{3}$
(b) 3
(c) $\frac{1}{2}$
(d) $\frac{1}{6}$
(e) 1

(correct)

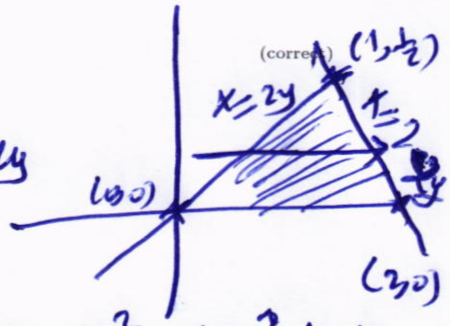
$$V = \int_0^{1/2}\int_{2y}^{2-2y} (2 - 2y - x)\, dx\, dy$$

$$= \int_0^{1/2} \left((2-2y)x - \frac{x^2}{2}\right)\Big|_{2y}^{2-2y} dy$$

$$= \int_0^{1/2} \left((2-2y)(2-2y-2y) - \frac{(2-2y)^2-(2y)^2}{2}\right) dy$$

$$= \int_0^{1/2} (8y^2 - 8y + 2)\, dy$$

$$= \left(\frac{8y^3}{3} - 4y^2 + 2y\right)\Big|_0^{1/2} = \frac{8}{3}\cdot\frac{1}{8} - 4\cdot\frac{1}{4} + 2\cdot\frac{1}{2} = \frac{1}{3}$$



16. The volume of the solid enclosed by the cone $z = \sqrt{x^2 + y^2}$, the cylinder $x^2 + y^2 = 9$ and the xy–plane is

$z = r$ $r = 3$ $z = 0$

(a) 18π
(b) 9π
(c) 6π
(d) 3π
(e) π

(correct)

Jacobian

$$V = \int_0^{2\pi}\int_0^3\int_0^r dz\, \boxed{r}\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^3 r^2\, dr\, d\theta$$

$$= \int_0^{2\pi} \left(\frac{r^3}{3}\Big|_0^3\right) d\theta$$

$$= \int_0^{2\pi} 9\, d\theta = 9\left(\theta\Big|_0^{2\pi}\right)$$

$$= 18\pi$$

17. The volume of the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the $xy-$plane is

(a) $8\,\pi$

(b) $6\,\pi$

(c) $4\,\pi$

(d) $2\,\pi$

(e) π

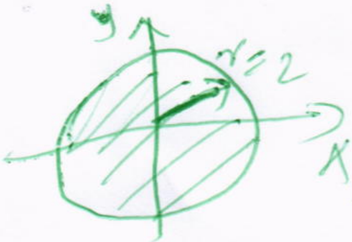
Jacobian

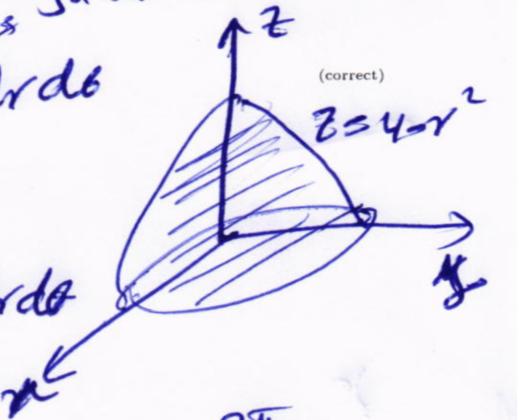
$$V = \int_0^{2\pi}\int_0^{2}\int_0^{4-r^2} dz\,\boxed{r}\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2} (4r - r^3)\,dr\,d\theta$$

$$= \int_0^{2\pi}\left(\left(2r^2 - \frac{r^4}{4}\right)\Big|_0^2\right)d\theta = \int_0^{2\pi} 4\,d\theta$$

$$= (4\,\theta)\Big|_0^{2\pi} = 8\pi.$$





18. The volume of the solid E that lies above the cone $\sqrt{3}z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=1$ is

(a) $\frac{\pi}{3}$

(b) $\frac{\pi}{6}$

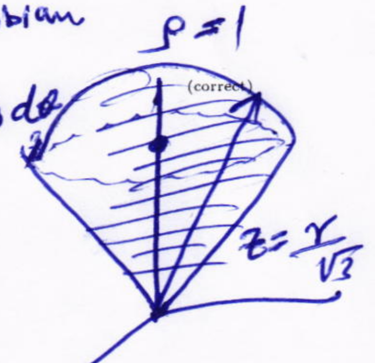
(c) $\frac{\pi}{4}$

(d) π

(e) $\frac{\pi}{2}$

Jacobian

$$V = \int_0^{2\pi}\int_0^{\pi/3}\int_0^{1} \boxed{\rho^2\sin\phi}\,d\rho\,d\phi\,d\theta$$



$$= \int_0^{2\pi}\int_0^{\pi/3}\left(\frac{\rho^3}{3}\Big|_0^1\right)\sin\phi\,d\phi\,d\theta$$

$\tan\phi = \frac{r}{z} = \frac{\sqrt{3}z}{z} = \sqrt{3}.$

$0 \le \phi \le \frac{\pi}{3}.$

$$= \frac{1}{3}\int_0^{2\pi}(-\cos\phi)\Big|_0^{\pi/3}d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\left(-\frac{1}{2}+1\right)d\theta = \frac{1}{3}\cdot\frac{1}{2}\int_0^{2\pi}d\theta$$

$$= \frac{1}{3}\cdot\frac{1}{2}(2\pi) = \frac{\pi}{3}$$

19. If E is the solid that lies within the cylinder $x^2 + y^2 = 1$, above the plane $z = 0$ and below the cone $z^2 = 4x^2 + 4y^2$, then $\iiint_E x^2\,dV = I$

(a) $\frac{2\pi}{5}$ (correct)

(b) $\frac{\pi}{4}$

(c) $\frac{\pi}{2}$

(d) $\frac{3\pi}{2}$

(e) $\frac{4\pi}{5}$

Jacobian

$$I = \int_0^{2\pi}\int_0^1\int_0^{2r} \boxed{r^2\cos^2\theta}\cdot r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 r^3\cdot 2r\,\cos^2\theta\,dr\,d\theta$$

$$= 2\int_0^{2\pi} \frac{r^5}{5}\Big|_0^1 \left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \frac{1}{5}\cdot\left(\theta + \frac{\sin(2\theta)}{2}\right)\Big|_0^{2\pi} = \frac{2\pi}{5}$$

20. If E is the region in the **first octant** that lies between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ and the coordinate planes, then

$$\iiint_E (x^2 + y^2)\,z\,dV =$$

(a) $\frac{21\pi}{16}$ (correct)

(b) $\frac{29\pi}{30}$

(c) $\frac{16\pi}{15}$

(d) $\frac{11\pi}{10}$

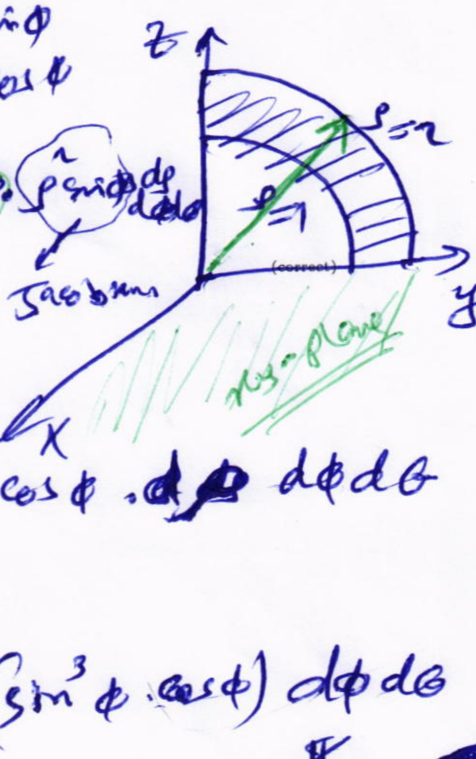
(e) $\frac{14\pi}{15}$

$r = \rho\sin\phi$

$z = \rho\cos\phi$

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 \rho^2\sin^2\phi\cdot\rho\cos\phi\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

Jacobian



$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 \rho^5\,\sin^3\phi\cdot\cos\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\left(\frac{\rho^6}{6}\Big|_1^2\right)\left(\sin^3\phi\cdot\cos\phi\right)d\phi\,d\theta$$

$$= \frac{64-1}{6}\cdot\frac{\sin^4(\phi)}{4}\Big|_0^{\pi/2}\cdot\theta\Big|_0^{\pi/2}$$

$$= \frac{63}{6}\cdot\frac{1}{4}\cdot\frac{\pi}{2} = \frac{21\pi}{16}$$

21. The number of **unit vectors** perpendicular to $\vec{u} = < 1, 0, -1 >$ is

(a) ∞ (correct)
(b) 0
(c) 1
(d) 2
(e) 3

Let $\vec{v} = <a, b, c> \perp \vec{u}$.

$\vec{v} \cdot \vec{u} = 0$

So, $a + 0 - c = 0$.

$$\boxed{a = c}$$

Since $|\vec{v}| = 1$; $a^2 + b^2 + c^2 = 1$.

So, $2a^2 + b^2 = 1$.

$$b = \mp \sqrt{1 - 2a^2}.$$

So, we can choose a such that

$1 - 2a^2 \geq 0$.

$a^2 \leq \frac{1}{2}$.

$a \in [-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$

& $b = \mp \sqrt{1 - 2a^2}$; $c = a$

The number of such vectors is ∞.

Geometrically, choose any plane containing $\vec{u}$. Any unit vector normal to this plane can be choosen. So, we have ∞ such vectors