King Fahd University of Petroleum and Minerals
Department of Mathematics

Math 201

Major Exam 1

212

February 23, 2022

Net Time Allowed: 90 Minutes

MASTER VERSION

Key Solution

1. The parametric curve

$$x = \frac{1}{\sqrt{t+1}} \text{ and } y = \frac{t}{t+1}, \ t > -1$$

represents:

(a) the right branch of the parabola $y = 1 - x^2$ excluding the vertex
(b) the parabola $y = x^2$ excluding the vertex
(c) the parabola $x = 1 - y^2$
(d) the upper branch of the parabola $x = 1 - y^2$ excluding the vertex
(e) the parabola $y = 1 - x^2$

$x^2 = \frac{1}{t+1} \Rightarrow t = \frac{1-x^2}{x^2}$

$\Rightarrow y = \frac{t}{t+1} \Rightarrow \boxed{y = 1-x^2} \ x \neq 0$

$t > -1 \Rightarrow \boxed{x > 0} \to$ right branch

2. If the position of two particles at time t ($0 \leq t \leq 2\pi$) is given by

$P_1: \ x_1 = 3 \sin t, \qquad y_1 = 2 \cos t$

$P_2: \ x_2 = -3 + \cos t, \qquad y_2 = 1 + \sin t$

set $x_1 = x_2$ and $y_1 = y_2$

Then the collision point of the two particles is

(a) $(-3, 0)$
(b) $(0, 2)$
(c) $(-1, 2)$
(d) $(3, 2)$
(e) $(-1, -3)$

$3 \sin t = -3 + \cos t$ — (1)

$2 \cos t = 1 + \sin t$ — (2)

From (2): $3 \sin t = 6 \cos t - 3$

Plug in (1): $6 \cos t - 3 = -3 + \cos t$

$\Rightarrow \boxed{\cos t = 0} \Rightarrow t = \frac{\pi}{2}, 3\frac{\pi}{2}$

If $t = \frac{\pi}{2}$: eq^n (1) $\Rightarrow 3 = -3$ (false)

However: $\boxed{t = 3\frac{\pi}{2}}$ satisfies both (1) & (2)

$\Rightarrow t = 3\frac{\pi}{2} \Rightarrow$ Collision point is $(-3, 0)$

3. Consider the parametric curve

$$C: \ x = 1 + t^2 \ , \ y = \frac{1}{t^2}.$$

An equation of the tangent line to C at the point $P(2,1)$ is

(a) $y = -x + 3$
(b) $y = x + 2$
(c) $y = -x - 1$
(d) $y = x - 1$
(e) $y = x - 3$

$x = 2 \Rightarrow 1 + t^2 = 2 \Rightarrow t^2 = 1 \Rightarrow t = \pm 1$

$\frac{\partial x}{\partial t} = 2t$ and $\frac{\partial y}{\partial t} = -\frac{2}{t^3}$

$\Rightarrow \boxed{\frac{dy}{dx} = \frac{-1}{t^4}}$

$\left.\frac{dy}{dx}\right|_{t=\pm 1} = -1 \Rightarrow y - 1 = -(x-2)$

$\Rightarrow \boxed{y = -x + 3}$

4. The area enclosed by the curve

$$x = t^4, \ y = t - t^2$$

and the x-axis is

(a) $\frac{2}{15}$
(b) $\frac{3}{17}$
(c) $\frac{4}{15}$
(d) $\frac{1}{5}$
(e) $\frac{1}{17}$

x-intercepts: $y = 0$

$t(1-t) = 0 \Rightarrow t = 0$ or $t = 1$

$\Rightarrow \text{Area} = \int_0^1 y \cdot \left(\frac{dx}{dt}\right) dt$

$= \int_0^1 (t - t^2) \cdot 4t^3 \cdot dt$

$= \int_0^1 4(t^4 - t^5)\, dt$

$= 4\left[\frac{t^5}{5} - \frac{t^6}{6}\right]_0^1$

$= 4\left(\frac{1}{5} - \frac{1}{6}\right) = 4 \cdot \frac{1}{30} = \frac{2}{15}$

5. The graph of the polar curve $r = \sin\theta$, where $0 \leq \theta < \pi$ has a horizontal tangent line when $(r, \theta) =$

(a) $\left(1, \frac{\pi}{2}\right)$

(b) $\left(\sqrt{2}, \frac{\pi}{4}\right)$

(c) $\left(\sqrt{2}, \frac{3\pi}{4}\right)$

(d) $\left(2\sqrt{2}, \frac{\pi}{4}\right)$

(e) $\left(2, \frac{\pi}{2}\right)$

$$\frac{dy}{dx} = \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}$$

$$\frac{dy}{dx} = 0 \Rightarrow \sin 2\theta = 0$$

$$\Rightarrow 2\theta = 0, \pi \Rightarrow \boxed{\theta = 0, \frac{\pi}{2}}$$

$$\theta = 0 \Rightarrow (0, 0)$$

$$\theta = \frac{\pi}{2} \Rightarrow \left(1, \frac{\pi}{2}\right)$$

6. The length of the cardioid

$$r = a(1 + \cos\theta), \quad a > 0$$

is

(a) $8a$

(b) a

(c) 0

(d) $4a$

(e) $\frac{a}{2}$

$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{a^2(1+\cos\theta)^2 + (-a\sin\theta)^2}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{2a^2(1+\cos\theta)}\, d\theta \qquad \boxed{a>0}$$

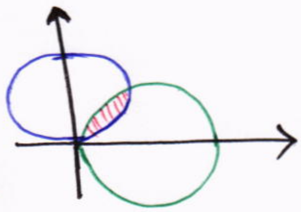
$$= 4a\int_0^{\pi} \cos\frac{\theta}{2}\, d\theta$$

$$= 8a\left[\sin\frac{\theta}{2}\right]_0^{\pi} = 8a$$

7. The area of the region that lies inside both curves $r = 3\sin\theta$ and $r = 3\cos\theta$ is

$3\sin\theta = 3\cos\theta$

$\Rightarrow \theta = \frac{\pi}{4}$



(a) $\dfrac{9\pi - 18}{8}$

(b) $\dfrac{\pi + 8}{8}$

(c) $\dfrac{\pi}{4}$

(d) 4π

(e) $\dfrac{\pi - 9}{8}$

$$A = \frac{1}{2}\int_0^{\pi/4} 9\sin^2\theta \, d\theta + \frac{1}{2}\int_{\pi/4}^{\pi/2} 9\cos^2\theta \, d\theta$$

By symmetry: $A = 2 \cdot \left(\frac{1}{2}\int_0^{\pi/4} 9\sin^2\theta \, d\theta\right)$

$$A = \frac{9}{2}\int_0^{\pi/4} (1 - \cos 2\theta)\, d\theta$$

$$= \frac{9}{2}\left[\theta - \frac{\sin(2\theta)}{2}\right]_0^{\pi/4} = \frac{9}{2}\left(\frac{\pi}{4} - \frac{1}{2}\right) = \frac{9\pi}{8} - \frac{9}{4}$$

$$= \boxed{\frac{9\pi - 18}{8}}$$

8. The equation $x^2 + y^2 + z^2 - 2y - 4z = -5$ represents

$x^2 + (y^2 - 2y + 1) + (z^2 - 4z + 4) = 0$

(a) the point $(0, 1, 2)$

(b) a sphere with center $(2, 0, -1)$

(c) a sphere with center $(0, 1, -2)$

(d) a sphere with radius $\sqrt{5}$

(e) no graph

$x^2 + (y-1)^2 + (z-2)^2 = 0$

This eq^n represents the point $(0,1,2)$.

9. The distance from the origin to the center of the sphere $x^2 + y^2 + z^2 = 4x + 4y + 4z - 11$ is

(a) $2\sqrt{3}$

(b) $\sqrt{5}$

(c) $\dfrac{1}{\sqrt{3}}$

(d) 4

(e) $\dfrac{2}{\sqrt{5}}$

$$(x^2 - 4x + 4) + (y^2 - 4y + 4) + (z^2 - 4z + 4) = 1$$

$$(x-2)^2 + (y-2)^2 + (z-2)^2 = 1$$

$\Rightarrow$ center is $(2,2,2)$

$$\Rightarrow d = \sqrt{(2-0)^2 + (2-0)^2 + (2-0)^2}$$

$$= \sqrt{12} = 2\sqrt{3}$$

10. Consider the points $P(-4, 1, -3)$ and $Q(0, 6, -5)$. If R is a point such that $-\frac{1}{2}\overrightarrow{PR} = \overrightarrow{PQ}$, then the magnitude of the position vector $\overrightarrow{OR}$ is

(a) $\sqrt{226}$

(b) $\sqrt{136}$

(c) $\sqrt{133}$

(d) $\sqrt{233}$

(e) $\sqrt{131}$

Let $R = (r_1, r_2, r_3)$

$$\Rightarrow -\frac{1}{2}\overrightarrow{PR} = \left\langle -\frac{(r_1+4)}{2}, -\frac{(r_2-1)}{2}, -\frac{(r_3+3)}{2} \right\rangle$$

$$\overrightarrow{PQ} = \langle 4, 5, -2 \rangle$$

So: $-\frac{1}{2}(r_1+4) = 4 \Rightarrow r_1 = -12$

$-\frac{1}{2}(r_2-1) = 5 \Rightarrow r_2 = -9$

$-\frac{1}{2}(r_3+3) = -2 \Rightarrow r_3 = 1$

$\Rightarrow R = (-12, -9, 1)$

$\Rightarrow |\overrightarrow{OR}| = \sqrt{144+81+1} = \sqrt{226}$

11. The unit vector in the direction of $2\vec{u}+\vec{v}$, where $\vec{u}=\langle 1,0,-1\rangle$ and $\vec{v}=\langle 2,-1,0\rangle$ is

(a) $\left\langle \frac{4}{\sqrt{21}}, \frac{-1}{\sqrt{21}}, \frac{-2}{\sqrt{21}} \right\rangle$

(b) $\left\langle \frac{3}{\sqrt{21}}, \frac{1}{\sqrt{21}}, \frac{1}{\sqrt{21}} \right\rangle$

(c) $\left\langle \frac{-4}{\sqrt{21}}, \frac{2}{\sqrt{21}}, \frac{2}{\sqrt{21}} \right\rangle$

(d) $\left\langle \frac{4}{\sqrt{21}}, \frac{1}{\sqrt{21}}, \frac{2}{\sqrt{21}} \right\rangle$

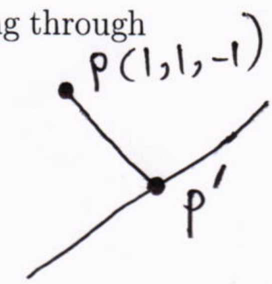
(e) $\left\langle \frac{4}{\sqrt{21}}, \frac{-1}{\sqrt{21}}, \frac{-1}{\sqrt{21}} \right\rangle$

$$2\vec{u}+\vec{v} = < 4, -1, -2 >$$

$$|2\vec{u}+\vec{v}| = \sqrt{16+1+4} = \sqrt{21}$$

$$\Rightarrow \frac{2\vec{u}+\vec{v}}{|2\vec{u}+\vec{v}|} = < \frac{4}{\sqrt{21}}, \frac{-1}{\sqrt{21}}, \frac{-2}{\sqrt{21}} >$$

12. If d is the distance from the point $P(1,1,-1)$ to the line L passing through the points $A(-1,0,0)$ and $B(0,1,1)$, then d^2 is



$$\vec{v} = \overrightarrow{AB} = <1,1,1>$$

(a) $\frac{14}{3}$

(b) $\frac{13}{2}$

(c) $\frac{11}{4}$

(d) $\frac{10}{3}$

(e) $\frac{2}{3}$

$$L: \ x = y-1 = z-1 = (t)$$

$$\Rightarrow P' = (t, t+1, t+1) = (x,y,z)$$

$$\Rightarrow \overrightarrow{PP'} = < t-1, t, t+2 >$$

choose P' so that $\overrightarrow{PP'} \perp L$:

$$(t-1) + t + (t+2) = 0 \Rightarrow \boxed{t = -\frac{1}{3}}$$

$$\Rightarrow \overrightarrow{PP'} = < -\frac{4}{3}, -\frac{1}{3}, \frac{5}{3} >$$

$$\Rightarrow d^2 = |\overrightarrow{PP'}|^2 = \frac{16}{9} + \frac{1}{9} + \frac{25}{9} = \frac{42}{9}$$

$$= \frac{14}{3}$$

13. Consider the points $A(2,1,-1)$, $B(3,0,2)$, $C(4,-2,1)$ and $D(1,m,n)$. The values of the real parameters m and n such that the vector $\overrightarrow{AD}$ is perpendicular to the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are related as

$\overrightarrow{AD} = \langle -1, m-1, n+1 \rangle$

$\overrightarrow{AB} = \langle 1, -1, 3 \rangle$

$\overrightarrow{AC} = \langle 2, -3, 2 \rangle$

(a) $3n = m - 3$ and $3m = 2n + 3$

(b) $3n = m + 4$ and $3m = 2n - 4$

(c) $3n = m + 3$ and $3m = 2n - 5$

(d) $3n = m - 1$ and $3m = 2n + 3$

(e) $3n = m + 2$ and $3m = 2n - 6$

$\overrightarrow{AD} \cdot \overrightarrow{AB} = -\cancel{1} - m + \cancel{1} + 3n + 3 = 0$

$\Rightarrow \boxed{3n = m - 3}$

& $\overrightarrow{AD} \cdot \overrightarrow{AC} = -\cancel{2} - 3m + 3 + 2n + \cancel{2} = 0$

$\Rightarrow \boxed{3m = 2n + 3}$

14. Let $\vec{u} = \langle 1, 2, 4 \rangle$ and $\vec{v} = \langle -1, 3, 1 \rangle$. A vector that is orthogonal to both $\vec{u}$ and $\vec{v}$ is

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 4 \\ -1 & 3 & 1 \end{vmatrix}$$

(a) $\langle 2, 1, -1 \rangle$

(b) $\langle 1, 1, 1 \rangle$

(c) $\langle 2, -1, 2 \rangle$

(d) $\langle -2, -1, -2 \rangle$

(e) $\langle -2, 2, 1 \rangle$

$= -10\vec{i} - 5\vec{j} + 5\vec{k}$

$= \langle -10, -5, 5 \rangle$

$= -5 \underline{\underline{\langle 2, 1, -1 \rangle}}$

15. The volume of the parallelepiped determined by the vectors $\vec{a} = \langle 1, 2, 3\rangle$, $\vec{b} = \langle -1, 1, 2\rangle$ and $\vec{c} = \langle 2, 1, 4\rangle$ is

(a) 9

(b) 6

(c) 3

(d) 4

(e) 7

$$V = a \cdot (b \times c)$$

$$= \begin{vmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 2 & 1 & 4 \end{vmatrix}$$

$$= 2 - 2(-8) + 3(-3)$$

$$= 2 + 16 - 9 = \underline{9}$$