1. The parametric curve

$$x = \sin(t),\ y = \sin^2(t),\ -\frac{\pi}{3} \le t \le \frac{8\pi}{3}$$

passes through the origin

$x = 0 = y.$

(a) 3 times
(b) 6 times
(c) 5 times
(d) 4 times
(e) 2 times

$\sin(t) = 0 \iff t = n\pi,\ n \in \mathbb{Z}$ (correct)

In the given interval:

$\boxed{t = 0,\ t = \pi,\ t = 2\pi}$

For all these values $y = \sin^2(t) = 0.$

2. A cartesian equation of the parametric curve

$$x = \sin(t),\ y = \cot^2(t),\ \frac{\pi}{2} \le t < \pi$$

is given by

$\cot^2(t) + 1 = \csc^2(t) = \dfrac{1}{\sin^2(t)}$

(a) $x^2(y+1) = 1,\ 0 < x \le 1$
(b) $y = x + \dfrac{1}{x},\ 0 < x \le 1$
(c) $x^2 y = 1 - x^2,\ -1 < x \le 0$
(d) $xy = 1 - \dfrac{1}{x},\ 0 < x \le 1$
(e) $y = \sqrt{x}(1-x),\ \dfrac{1}{2} < x \le 1$

$y + 1 = \dfrac{1}{x^2}$ (correct)

$x^2(y+1) = 1.$

$\frac{\pi}{2} \le t < \pi$

$0 < \sin(t) \le 1$

$0 < x \le 1$

3. The length of the parametric curve

$$x = \cos(t) + t\sin(t),\ y = \sin(t) - t\cos(t),\ t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

is

(a) $\frac{\pi^2}{4}$

(b) $\frac{\pi}{3}$

(c) $\frac{3}{\pi^2}$

(d) π

(e) $\frac{2\pi}{9}$

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= \int_{-\pi/2}^{\pi/2} \sqrt{(\cancel{-\sin t} + \cancel{\sin t} + t\cos t)^2 + (\cancel{\cos t} - \cancel{\cos t} + t\sin t)^2}\, dt$$

(correct)

$$= \int_{-\pi/2}^{\pi/2} \sqrt{t^2(\cos^2(t) + \sin^2(t))}\, dt$$

$$= \int_{-\pi/2}^{\pi/2} \sqrt{t^2}\, dt = \int_{-\pi/2}^{\pi/2} |t|\, dt = 2\int_0^{\pi/2} t\, dt$$

$$= \cancel{2} \cdot \frac{t^2}{\cancel{2}}\Big|_0^{\pi/2}$$

$$= \frac{\pi^2}{4}$$

4. The parametric curve

$$x = t - \ln(t),\ y = t + \ln(t)$$

is concave up on

(a) $(0, 1)$

(b) $(-\infty, 0) \cup (1, \infty)$

(c) $(1, \infty)$

(d) $(-\infty, 1)$

(e) $(-1, 0)$

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 + \frac{1}{t}}{1 - \frac{1}{t}} = \frac{t+1}{t-1}$$

$$\frac{d^2y}{dx^2} = \frac{d(y')/dt}{dx/dt} = \frac{\frac{1\cancel{(t-1)} - 1\cancel{(t+1)}}{(t-1)^2}}{\frac{(t-1)}{t}}$$

(correct)

$$= \frac{-2t}{(t-1)^3}$$

y'': − − − (0) + + + + (1, D.E) − − − −

5. The graph of the polar equation

$$r = 2(\cos(\theta) - \sin(\theta))$$

is a circle with

(a) center $(1, -1)$ and radius $\sqrt{2}$
(b) center $(-1, 1)$ and radius $\sqrt{2}$
(c) center $(1, 1)$ and radius $\sqrt{2}$
(d) center $(-1, -1)$ and radius $\sqrt{2}$
(e) center $(-1, -1)$ and radius 2

$r^2 = 2(r\cos(\theta) - r\sin(\theta))$

$x^2 + y^2 = 2x - 2y.$

$(x-1)^2 + (y+1)^2 = 2.$

(correct)

center $(1, -1)$

radius $\sqrt{2}$.

6. The slope of the tangent line to the curve

$$r = 1 + 2\sin(\theta)\cos(\theta) = 1 + \sin(2\theta)$$

at $\theta = \dfrac{\pi}{4}$ is

(a) -1
(b) 1
(c) 0
(d) 2
(e) -2

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

(correct)

$$= \frac{\cancel{2\cos(2\theta)}^{\text{zero}} \cdot \sin\theta + (1+\sin(2\theta))\cos\theta}{\cancel{2\cos(2\theta)}_{\text{zero}}\cos\theta - (1+\sin(2\theta))\sin\theta}$$

$$= -\cot(\theta)$$

$$m_{\text{tangent}} = \left.\frac{dy}{dx}\right|_{\theta = \frac{\pi}{4}} = -\cot(\pi/4) = -1.$$

7. The length of the polar curve

$$r = \frac{e^{2\theta}}{\sqrt{5}},\ 0 \le \theta \le \pi$$

is

(a) $\dfrac{e^{2\pi}-1}{2}$

(b) $e^{2\pi}-1$

(c) $\dfrac{e^{2\pi-1}}{5}$

(d) $e^{2\pi}-5$

(e) $e^{2\pi+1}$

$\left(\frac{dr}{d\theta}\right)^2 = \left(\frac{2e^{2\theta}}{\sqrt{5}}\right)^2 = \frac{4}{5}e^{4\theta}$

$r^2 = \left(\frac{e^{2\theta}}{\sqrt{5}}\right)^2 = \frac{e^{4\theta}}{5}$

(correct)

$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

$= \int_0^\pi \sqrt{\frac{e^{4\theta}}{5} + \frac{4e^{4\theta}}{5}}\, d\theta$

$= \int_0^\pi e^{2\theta}\, d\theta = \frac{e^{2\theta}}{2}\Big|_0^\pi$

$= \frac{e^{2\pi}-1}{2}$

8. The area of the region bounded by the circle $r = 2\sin(\theta)$ for $\frac{\pi}{12} \le \theta \le \frac{\pi}{2}$ is

$r^2 = 2r\sin\theta \qquad x^2+y^2=2y$

$x^2+(y-1)^2=1$

(a) $\dfrac{5\pi+3}{12}$

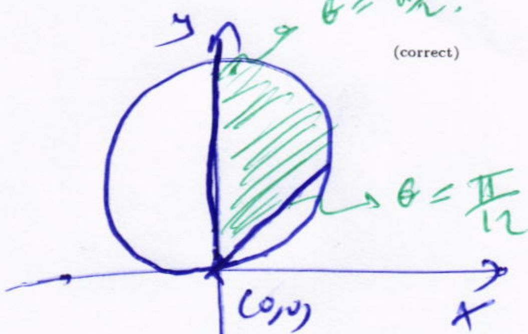
(b) $\dfrac{5\pi-3}{12}$

(c) $\dfrac{5\pi-3\sqrt{3}}{12}$

(d) $\dfrac{5\pi+3\sqrt{3}}{12}$

(e) $\dfrac{5\pi}{12}$

(correct)



$A = \frac{1}{2}\int_\alpha^\beta r^2\, d\theta = \frac{1}{2}\int_{\pi/12}^{\pi/2} (2\sin\theta)^2\, d\theta$

$= 2\int_{\pi/12}^{\pi/2} \sin^2\theta\, d\theta = \cancel{2}\int_{\pi/12}^{\pi/2} \frac{(1-\cos(2\theta))}{\cancel{2}}\, d\theta$

$= \left(\theta - \frac{\sin(2\theta)}{2}\right)\Big|_{\pi/12}^{\pi/2} = \left(\frac{\pi}{2} - \frac{\pi}{12}\right) - \frac{(\overset{\text{zero}}{\sin(\pi)} - \sin(\frac{\pi}{6}))}{2}$

$= \frac{5\pi}{12} + \frac{1}{4} = \boxed{\frac{5\pi+3}{12}}.$

9. Let C be the circle of intersection between the sphere

$$x^2 + y^2 + \cancel{z^2} + mx + 2y + \cancel{2z} + 9 = 0$$

zero zero

and the $\boxed{xy\text{-plane}}$. If the radius of this circle is 2, then one of the following is a possible center for C :

zero.

(a) $(-2\sqrt{3}, -1)$ (correct)
(b) $(\sqrt{3}, -1)$
(c) $(2\sqrt{2}, -1)$
(d) $(-2\sqrt{3}, 1)$
(e) $(-\sqrt{3}, -1)$

$$(x+\frac{m}{2})^2 + (y+1)^2 + 9 = \frac{m^2}{4} + 1$$

$$(x+\frac{m}{2})^2 + (y+1)^2 = \frac{m^2}{4} - 8.$$

$$\frac{m^2}{4} - 8 = 4.$$

$$\frac{m^2}{4} = 12; \quad \boxed{m^2 = 48}$$

$$m = \mp\sqrt{48} = \mp 4\sqrt{3}.$$

$\boxed{(-\frac{m}{2}, -1) \text{ Possible centers}}$: $(2\sqrt{3}, -1)$ & $(-2\sqrt{3}, -1)$.

10. The vector $\vec{v}$ in the xy-plane with length 2 and making an angle of $\frac{\pi}{6}$ with the positive x-axis is:

(a) $\langle\sqrt{3}, 1\rangle$ (correct)
(b) $\langle-\sqrt{3}, 1\rangle$
(c) $\langle\sqrt{2}, \sqrt{2}\rangle$
(d) $\langle-\sqrt{2}, \sqrt{2}\rangle$
(e) $\langle\sqrt{2}, -\sqrt{2}\rangle$

Let $\vec{v} = \langle a, b\rangle$

$$\frac{b}{a} = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}.$$

$\boxed{\text{or}}$ $\frac{b}{a} = \tan(-\frac{\pi}{6}) = -\frac{1}{\sqrt{3}}$

I $\boxed{a = \sqrt{3}\, b.}$

$|\vec{v}| = 2.$

$a^2 + b^2 = 4.$

$3b^2 + b^2 = 4.$

$4b^2 = 4.$

$b^2 = 1$

$b = \mp 1.$

II $a = -\sqrt{3}\, b$

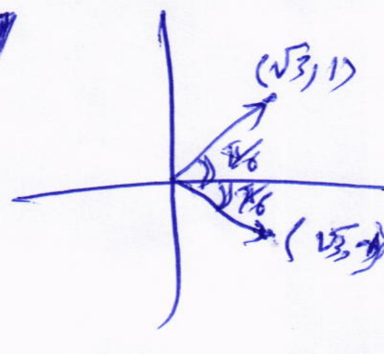
$\boxed{\langle -\sqrt{3}, +1\rangle}$

$\langle +\sqrt{3}, 1\rangle$

Two possible vectors

$\boxed{\langle\sqrt{3}, 1\rangle}$

$\langle -\sqrt{3}, -1\rangle$



11. If the vector $\vec{v} = \langle k, k+1 \rangle$ is parallel to the tangent line of the curve of $y = 1 + x + \sin(x)$ at the point $(0, 1)$, then the value of k is

(a) 1 (correct)

(b) 2

(c) -1

(d) -2

(e) $\frac{1}{2}$

$y' = 1 + \cos(x)$.

$m_{\text{tangent}} = y'(0) = 2$.

$\therefore \frac{k+1}{k} = 2$.

$k + 1 = 2k$.

$\boxed{k = 1}$

12. The set of **all** possible values of b such that vectors $\langle -6, b, 2 \rangle$ and $\langle b, b^2, b \rangle$ are orthogonal is

(a) $\{-2, 0, 2\}$ (correct)

(b) $\{-2, 2\}$

(c) $\{-1, 0, 1\}$

(d) $\{-1, 1\}$

(e) $\{0\}$

$\vec{u} \perp \vec{v} \overset{\text{Definition}}{\Leftrightarrow} \vec{u} \cdot \vec{v} = 0$.

So, we need to have

$-6b + b^3 + 2b = 0$.

$b^3 - 4b = 0$.

$b(b^2 - 4) = 0$.

$b(b-2)(b+2) = 0$.

$\therefore b = 0, \; b = 2, \; b = -2$.

Note The zero vector is orthogonal to any vector.

13. If θ is the angle between the diagonal of the **unit cube** (in the first octant with one corner at the origin and three edges along the coordinate axes) and one of these edges, then the value of $\cos(\theta)$ is

(a) $\frac{1}{\sqrt{3}}$ (correct)
(b) $\frac{1}{\sqrt{2}}$
(c) $\frac{1}{2}$
(d) $\frac{\sqrt{3}}{2}$
(e) $\frac{2}{\sqrt{6}}$

We can obtain θ as the angle between the vectors $\vec{u} = \langle 1,0,0\rangle$ & $\vec{v} = \langle 1,1,1\rangle$.

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \frac{1+0+0}{(1)(\sqrt{3})} = \frac{1}{\sqrt{3}}.$$

14. The area of the triangle whose vertices are $P(0,-1,1)$, $Q(-1,1,2)$ and $R(2,1,-1)$ is

(a) $3\sqrt{2}$ (correct)
(b) $6\sqrt{2}$
(c) $5\sqrt{2}$
(d) $4\sqrt{2}$
(e) $2\sqrt{2}$

$\vec{PQ} = \langle -1, 2, 1\rangle$

$\vec{PR} = \langle 2, 2, -2\rangle$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 1 \\ 2 & 2 & -2 \end{vmatrix}$$

$$= \langle -6, 0, -6\rangle$$

$$\text{Area} = \frac{1}{2}|\vec{PQ}\times\vec{PR}| = \frac{1}{2}\sqrt{36+36} = 3\sqrt{2}.$$

15. A parallelepiped has adjacent sides PQ, PR and PS where $P(2,1,-1)$, $Q(3,0,2)$, $R(4,-2,1)$ and $S(5,c,0)$.
If the volume of this parallelepiped is 4, then the sum of all possible values of c is

$\vec{PQ} = \langle 1,-1,3 \rangle$, $\vec{PR} = \langle 2,-3,2 \rangle$, $\vec{PS} = \langle 3, c-1, 1 \rangle$

(a) -8 (correct)
(b) -6
(c) 8
(d) 6
(e) 4

$$V = \left| \begin{vmatrix} 1 & -1 & 3 \\ 2 & -3 & 2 \\ 3 & c-1 & 1 \end{vmatrix} \right| = \left| (2(c-1)+3) + (-4) + 3(2c-2+9) \right|$$

$$= |4c+16| = 4|c+4|$$

$V = 4 \iff |c+4| = 1 \iff c = -3$ or -5

$\iff (c+4) = +1$ or -1

$\therefore$ the sum is $(-3) + (-5) = -8$

16. The parametric curve

$$x = 2t - \pi\sin(t),\ y = 2 - \pi\cos(t),\ t \in [-\pi, \pi]$$

has at the point $(0,2)$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\pi\sin(t)}{2-\pi\cos(t)}$$

(a) two slant tangents (correct)
(b) one horizontal tangent
(c) one vertical tangent
(d) only one slant tangent with positive slope
(e) only one slant tangent with negative slope

$\boxed{x=0 \iff 2t - \pi\sin(t) = 0.}$

$y = 0 \iff \cancel{2} - \pi\cos(t) = \cancel{2}$

$\iff \cos(t) = 0.$

So $t = \mp\frac{\pi}{2}$

satisfy one

check

$$\left.\frac{dy}{dx}\right|_{t=-\frac{\pi}{2}} = \frac{\pi\sin(-\frac{\pi}{2})}{2-0} = -\frac{\pi}{2}$$

$$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{2}} = \frac{\pi\sin(\frac{\pi}{2})}{2-0} = \frac{\pi}{2}$$

two slant tangents

17. The area of the region that lies inside both circles $r = 1$ and $\boxed{r = 2\sin(\theta)}$ is

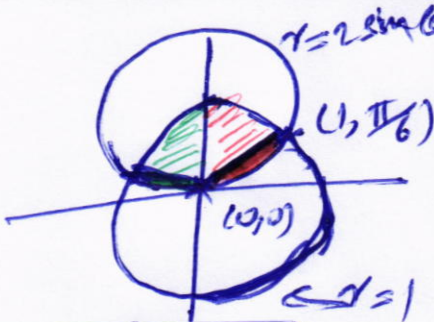
(a) $\dfrac{4\pi - 3\sqrt{3}}{6}$

(b) $\dfrac{4\pi - 3\sqrt{3}}{3}$

(c) $\dfrac{4\pi - 3}{6}$

(d) $\dfrac{4\pi - 3}{3}$

(e) $\dfrac{2\pi - 3}{3}$



$r^2 = 2r\sin\theta$
$x^2+y^2 = 2y$.
$x^2+(y-1)^2 = 1$. (correct)

$1 = 2\sin\theta$
$\sin\theta = \frac{1}{2}$

$\boxed{\text{Using symmetry}}$

$\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$.

$$A = \cancel{2}\cdot\frac{1}{\cancel{2}}\left(\int_0^{\pi/6}(2\sin\theta)^2\,d\theta + \int_{\pi/6}^{\pi/2}(1)^2\,d\theta\right)$$

$$A = 4\int_0^{\pi/6}\sin^2\theta\,d\theta + \theta\Big|_{\pi/6}^{\pi/2} = 4\cdot\int_0^{\pi/6}\frac{1-\cos(2\theta)}{2}\,d\theta + \left(\frac{\pi}{2}-\frac{\pi}{6}\right)$$

$$= 2\left(\theta - \frac{\sin(2\theta)}{2}\Big|_0^{\pi/6}\right) + \frac{2\pi}{6} = \frac{2\pi}{6} - \sin\left(\frac{\pi}{3}\right) + \frac{2\pi}{6}$$

$$= \frac{4\pi}{6} - \frac{\sqrt{3}}{2}$$

$$= \boxed{\frac{4\pi - 3\sqrt{3}}{6}}$$

18. If $\vec{u}$, $\vec{v}$ and $\vec{w}$ are three vectors such that

$$\vec{u}\cdot(\vec{v}\times\vec{w}) = 3,$$

then

$$(\vec{w}\times\vec{v})\cdot(\vec{u}-\vec{v}) = (\vec{w}\times\vec{v})\cdot\vec{u} - \cancel{(\vec{w}\times\vec{v})\cdot\vec{v}}$$

Zero

$(\vec{w}\times\vec{v}) \perp \vec{v}$, $\perp \vec{w}$ (correct)

$$= \vec{u}\cdot(\vec{w}\times\vec{v})$$

$$= \vec{u}\cdot(-)(\vec{v}\times\vec{w})$$

$$= -\boxed{(\vec{u})\cdot(\vec{v}\times\vec{w})}$$

$$= -(3) = -3.$$

(a) -3

(b) 3

(c) 0

(d) 6

(e) -6

19. The graph of the parametric curve

$$\boxed{x = 1 - 3t^4, y = 1 + 3t^4, 0 \le t < \infty}$$

is

$3t^4 = y - 1.$

(a) a straight line with slope -1 (correct)

(b) a parabola

(c) a straight line with slope 2

(d) a circle

(e) a hyperbola

$\therefore x = 1 - (y - 1)$

$x = 1 - y + 1$

$\boxed{y = 2 - x}$

straight line with slope -1.

20. If a vector $\vec{v}$ has direction angles $\alpha = \frac{\pi}{4}$ with the positive x-axis and $\beta = \frac{\pi}{3}$ with the positive y-axis, then the set of **all** possible values of the angle γ between $\vec{v}$ and the positive z-axis is

(a) $\left\{\frac{\pi}{3}, \frac{2\pi}{3}\right\}$ (correct)

(b) $\left\{\frac{\pi}{3}\right\}$

(c) $\left\{\frac{2\pi}{3}\right\}$

(d) $\left\{\frac{\pi}{4}\right\}$

(e) $\left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$

$\cos^2\left(\frac{\pi}{4}\right) + \cos^2\frac{\pi}{3} + \cos^2\gamma = 1.$

$\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 + \cos^2(\gamma) = 1.$

$\frac{1}{2} + \frac{1}{4} + \cos^2(\gamma) = 1.$

$\therefore \cos^2(\gamma) = \frac{1}{4}.$

$\cos(\gamma) = \mp\frac{1}{2}.$

equivalently $\gamma = \frac{\pi}{3}$ or $\frac{2\pi}{3}$

since $0 \le \gamma \le \pi$.