1. If the symmetric equations of the line through the point $(2, 3, -7)$ and orthogonal to the plane $x - y + 5z = 1$ are given by

$$\frac{x-2}{2} = \frac{y-3}{b} = \frac{z+7}{c},$$

then $b + c =$

(a) 8 (correct)
(b) 4
(c) -8
(d) -4
(e) 13

$L \parallel \langle 1, -1, 5 \rangle$

$$\frac{x-2}{1} = \frac{y-3}{-1} = \frac{z-(-7)}{5}$$

$$\therefore \frac{x-2}{2} = \frac{y-3}{-2} = \frac{z+7}{10}$$

$b + c = -2 + 10 = 8$

2. The distance between the two planes

$$x - 3y + 4z = 5 \text{ and } \boxed{x - 3y + 4z = 3}$$

is

(a) $\frac{2}{\sqrt{26}}$ (correct)
(b) $\frac{8}{\sqrt{26}}$
(c) $\frac{3}{\sqrt{13}}$
(d) $\frac{2}{5}$
(e) $\frac{1}{3}$

$Ax + By + Cz + d = 0.$
$x - 3y + 4z + (-3) = 0.$

The distance between the two parallel planes is the same as the distance between any point on the first plane and the second plane.

Consider $P(5, 0, 0)$ on the first plane.

$$d = \frac{|Ax_0 + By_0 + Cz_0 + d|}{\sqrt{A^2 + B^2 + C^2}}$$

$$= \frac{|1(5) + 0 + 0 + (-3)|}{\sqrt{(1)^2 + (-3)^2 + (4)^2}} = \frac{2}{\sqrt{26}}$$

3. The line passing through the point $(1, 2, 3)$ and orthogonal to the plane $x + 3y + z = 21$ passes also through the point

(a) $(0, -1, 2)$ (correct)

(b) $(2, 5, 0)$

(c) $(3, 0, 5)$

(d) $(-2, -7, 1)$

(e) $(-1, 1, 2)$

$L \parallel \langle 1, 3, 1 \rangle$.

$$\left.\begin{aligned} x &= x_0 + at = 1 + t \\ y &= y_0 + bt = 2 + 3t \\ z &= z_0 + ct = 3 + t \end{aligned}\right\} t \in \mathbb{R}.$$

The line passes through $(0, -1, 2)$ at $t = -1$.

No other point of the given ones satisfies the parametric equations of L for any $t \in \mathbb{R}$.

4. The intersection of the paraboloids $z = x^2 + y^2$ and $z = 4 - x^2 - y^2$ is

(a) a circle (correct)

(b) an ellipse that is not a circle

(c) a plane

(d) a straight line

(e) a single point

$$x^2 + y^2 = 4 - (x^2 + y^2)$$

$$2(x^2 + y^2) = 4.$$

$$\boxed{x^2 + y^2 = 2}$$

Circle in the plane $z = 2$ with center $(0, 0, 2)$ & radius $\sqrt{2}$.

5. The quadratic equation

$$-2x^2 + 8x + y + 5z^2 = 8$$

represents

$$-2(x^2 - 4x + 4) + y + 5z^2 = 8 - 8$$

(a) a hyperbolic paraboloid
(b) an elliptic paraboloid
(c) an elliptic cone
(d) a hyperboloid of one sheet
(e) a hyperboloid of two sheets

$$-2(x-2)^2 + y + 5z^2 = 0.$$

(correct)

$$y = 2(x-2)^2 - 5z^2.$$

$$\boxed{y = \frac{(x-2)^2}{\left(\frac{1}{\sqrt{2}}\right)^2} - \frac{z^2}{\left(\frac{1}{\sqrt{5}}\right)^2}}$$

6. Consider the following statements about the surface

$$x = \sqrt{4y^2 + z^2 - 8y + 2z + 9} \geq 0$$

✗ (I) Its graph consists of two sheets

✗ (II) It has a vertex at $(0, 1, -1)$ (does not satisfy the eqn.)

✓ (III) Its axis is parallel to the x-axis

Which of the statement(s) above is (are) true about this surface?

$x \geq 0$ & $x^2 = 4(y^2 - 2y + 1) + (z+1)^2 + 4$

(a) Only III
(b) II and III
(c) I, II and III
(d) I and III
(e) Only II

(correct)

$x \geq 0$ & $x^2 - \dfrac{(y-1)^2}{\left(\frac{1}{2}\right)^2} - (z+1)^2 = 4$

$$\boxed{\boxed{x \geq 0} \ \& \ \frac{x^2}{(2)^2} - (y-1)^2 - \frac{(z+1)^2}{4} = 1}$$

One sheet of the hyperboloid of two sheets with axis parallel to the x-axis.

7. The range of

$$f(x, y, z) = \frac{\sqrt{1 - z^2}}{4 + \sqrt{1 - (x^2 + y^2)}} \geq 0. \quad \text{Clear}$$

the min. is equal to 0 for $z = \mp 1$

is

(a) $\left[0, \frac{1}{4}\right]$

(b) $\left[0, \frac{1}{2}\right]$

(c) $[-1, 1]$

(d) $\left[0, \frac{1}{5}\right]$

(e) $[-1, 2]$

The max. is obtained when the numerator is max. & the denomerator is min.

(correct)

$N: \sqrt{1-z^2} \leq 1$ (max. when $z=0$).

$D: 4 + \sqrt{1-(x^2+y^2)} \geq 4$ (min. when $x^2+y^2=1$).

$\therefore \quad f(x,y,z) \leq \frac{\max(N)}{\min(D)} \leq \frac{1}{4}$.

8. $\lim_{(x,y)\to(0,0)} \frac{x^3 + y^3}{x^2 - xy + y^2}$

$$= \lim_{(x,y)\to(0,0)} \frac{(x+y)\cancel{(x^2-xy+y^2)}}{\cancel{x^2-xy+y^2}}$$

(correct)

(a) equals 0

(b) equals 1

(c) equals -1

(d) equals 2

(e) does not exist

$$= \lim_{(x,y)\to(0,0)} (x+y) = 0+0 = 0.$$

9. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^3+y^3+z^3}$

(a) does not exist
(b) equals 1
(c) equals -1
(d) equals 3
(e) equals 0

Path ① $x=0$; $y=0, z\neq 0$.

$L = \lim_{z\to 0} \frac{0}{0+0+z^3} \overset{\text{(correct)}}{=} 0$.

$\boxed{\text{Path 2}}$ $x=y=z$

$L = \lim_{x\to 0} \frac{x^3}{3x^3} = \frac{1}{3}$.

The limit is path dependent, whence D.N.E.

10. The value of c that makes that function

$$f(x,y) = \begin{cases} \dfrac{\tan(x^2+y^2)}{x^2+y^2}, & (x,y)\neq(0,0) \\ c, & (x,y)=(0,0) \end{cases}$$

continuous at $(0,0)$ is

(a) 1
(b) 0
(c) -1
(d) 2
(e) no such value exists

$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} \frac{\tan(r^2)}{r^2}$

$= \lim_{r\to 0} \left(\frac{\sin(r^{2})}{r^2} \cdot \frac{1}{\cos(r^2)} \right)$ (correct)

$= \lim_{u\to 0} \frac{\sin(u)}{u} \lim_{u\to 0} \frac{1}{\cos(u)}$

$= (1)(1) = 1$.

$\boxed{c=1}$ makes the function continuous at $(0,0)$.

11. If $f(x,y) = (\tan^{-1}(xy))^2$, then $f_y(1,1) =$

(a) $\frac{\pi}{4}$
(b) $\frac{\pi}{3}$
(c) $-\frac{\pi}{4}$
(d) $-\frac{\pi}{3}$
(e) 0

$f_y = 2(\tan^{-1}(xy)) \cdot \frac{1}{1+(xy)^2} \cdot x$ (correct)

$= \cancel{2} \tan^{-1}(1) \cdot \frac{1}{\cancel{1+1}} \cdot 1$

$= \frac{\pi}{4}.$

12. If

$$f(x,y,z) = \sqrt{\cos^2(x) + \sin^2(y) + \cos^2(z)},$$

then $f_x\left(\frac{\pi}{4}, 0, \frac{\pi}{2}\right) =$

(a) $-\frac{1}{\sqrt{2}}$
(b) $\frac{1}{\sqrt{2}}$
(c) $\sqrt{2}$
(d) $-\sqrt{2}$
(e) $-\frac{1}{2}$

$f_x(x,y) = \frac{2\cos(x)\cdot(-\sin(x)) + 0 + 0}{2\sqrt{\cos^2(x) + \sin^2(y) + \cos^2(z)}}$ (correct)

$f_x\left(\frac{\pi}{4}, 0, \frac{\pi}{2}\right) = \frac{\cancel{2}\left(\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right)}{2\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + 0 + 0}}$

$= \frac{-1}{2\cdot\frac{1}{\sqrt{2}}} = -\frac{1}{\sqrt{2}}.$

13. If $f(x,y) = y\sin^{-1}(xy)$, then

$$f_x(1,0) + f_y(1,0) =$$

(a) 0
(b) 1
(c) 2
(d) 3
(e) 4

(correct)

$$f_x(x,y) = y \cdot \frac{1}{\sqrt{1-(xy)^2}} \cdot y$$

$$f_y(x,y) = 1 \cdot \sin^{-1}(xy) + y \cdot \frac{1}{\sqrt{1-(xy)^2}} \cdot x$$

$$f_x(1,0) + f_y(1,0) = 0 + (0 + 0)$$

$$= 0$$

14. The linearization of $f(x,y) = x^2y + \sqrt{x^2+y^2}$ at the point $(1,0)$ is $L(x,y) =$

(a) $x + y$
(b) $2x - y + 1$
(c) $x + y + 1$
(d) $x - y + 1$
(e) $x - 1$

(correct)

$$f_x = 2xy + \frac{1}{\cancel{2}\sqrt{x^2+y^2}} \cdot \cancel{2}x$$

$$f_y = x^2 + \frac{1}{\cancel{2}\sqrt{x^2+y^2}} \cdot \cancel{2}y$$

$$f_x(1,0) = 0 + \frac{1}{1} = 1.$$

$$f_y(1,0) = 1 + 0 = 1.$$

$$f(1,0) = 0 + 1 = 1.$$

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$

$$= 1 + 1(x-1) + 1\cdot(y-0)$$

$$= \cancel{1} + x \cancel{-1} + y = x+y$$

15. If $z = x^2 - xy + 3y^2$ and (x, y) changes from P $(2, -1)$ to Q $(1.96, -0.95)$, then $dz =$

(a) -0.6 (correct)
(b) 1.6
(c) 0.6
(d) -1.6
(e) 0

$$\boxed{\frac{\partial z}{\partial x} = 2x - y} \; \& \; \boxed{\frac{\partial z}{\partial y} = -x + 6y}$$

$$\boxed{dz = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} \cdot dy}$$

$$\underset{P \to Q}{dz} = \left.\frac{\partial z}{\partial x}\right|_P \cdot dx + \frac{\partial z}{\partial y} \cdot dy$$

$$= (5) \cdot \left(\frac{-4}{100}\right) + (-8)\left(\frac{5}{100}\right)$$

$$= -\frac{20}{100} - \frac{40}{100} = \frac{-60}{100} = \frac{-6}{10}$$

$$\boxed{= -0.6}$$

16. Using the linearization of

$$f(x, y) = e^{x-y}$$

at $(2, 2)$, the approximate value of $f(2.01, 1.99)$ is

(a) 1.02 (correct)
(b) 1.01
(c) 0.02
(d) 0.01
(e) 1.03

$$f_x = e^{x-y} \cdot (1)$$

$$f_y = e^{x-y}(-1)$$

$$f_x(2,2) = 1.$$

$$f_y(2,2) = -1.$$

$$f(x,y) \approx L(x,y) = f(x_0,y_0) + f_x|_P (x-x_0) + f_y|_P (y-y_0)$$

$$= 1 + 1(x-2) + (-1)(y-2)$$

$$= x - y + 1.$$

$$f(2.01, 1.99) \approx L(2.01, 1.99) = 2.01 - 1.99 + 1 = 1.02$$

17. If

$$F(x, y, z) = x^3 e^{y+z} - y \sin(x - z) = e^2,$$

then the value of $\dfrac{\partial z}{\partial x}$ at the point $(1, 1, 1)$ is

(a) $\dfrac{1 - 3e^2}{1 + e^2}$ (correct)

(b) $\dfrac{3e^2 - 1}{e^2 + 1}$

(c) $\dfrac{3e^2 - 1}{e^2 - 1}$

(d) $\dfrac{1 + 3e^2}{1 - e^2}$

(e) $\dfrac{1 + 3e^2}{1 + e^2}$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

$$= -\frac{3x^2 e^{y+z} - y\cos(x-z)(1)}{x^3 e^{y+z}(1) - y\cos(x-z)(-1)}$$

$$\left.\frac{\partial z}{\partial x}\right|_{(1,1,1)} = -\frac{3e^2 - 1}{e^2 + 1} = \frac{1 - 3e^2}{1 + e^2}.$$

18. If $w = x^2 + y^2 + z^2 + xy$, where $x = st$, $y = s - t$ and $z = s + 2t$, then the value of $\dfrac{\partial w}{\partial t}$ at $(s, t) = (1, 1)$ is

$(x, y, z) = (1, 0, 3)$

(a) 13 (correct)

(b) 14

(c) 12

(d) 11

(e) 10

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial t}$$

$$= (2x + y)(s) + (2y + x)(-1) + (2z)(2)$$

$$= (2)(1) + (1)(-1) + (6)(2)$$

$$= 2 - 1 + 12$$

$$= 13.$$

19. The directional derivative of $f(x,y,z) = \dfrac{x}{y+z}$ at $P(1,-1,2)$ in the direction of $Q(2,0,-1)$ is

(a) $\dfrac{3}{\sqrt{11}}$ (correct)

(b) $\dfrac{6}{\sqrt{11}}$

(c) $-\dfrac{4}{\sqrt{11}}$

(d) $\dfrac{2}{\sqrt{11}}$

(e) $\dfrac{8}{\sqrt{11}}$

$\vec{u} = \vec{PQ} = \langle 1, 1, -3 \rangle$

$\vec{\nabla} f = \langle f_x, f_y, f_z \rangle$

$= \left\langle \dfrac{1}{y+z}, -\dfrac{x}{(y+z)^2}, \dfrac{-x}{(y+z)^2} \right\rangle$

$\vec{\nabla} f\big|_P = \langle 1, -1, -1 \rangle$

$D_{P,u} f = \vec{\nabla} f\big|_P \cdot \dfrac{\vec{u}}{|\vec{u}|} = \dfrac{\langle 1,-1,-1\rangle \cdot \langle 1,1,-3\rangle}{\sqrt{1+1+9}}$

$= \dfrac{1-1+3}{\sqrt{11}} = \dfrac{3}{\sqrt{11}}$

20. The maximum rate of change of $f(x,y,z) = 4z + x^2 e^{-y}$ at $P(6, \ln 2, 3)$ occurs in the direction of

(a) $\langle 3, -9, 2 \rangle$ (correct)

(b) $\langle 3, 9, -2 \rangle$

(c) $\langle -3, 9, -2 \rangle$

(d) $\langle -3, -9, -2 \rangle$

(e) $\langle 3, -9, -2 \rangle$

The max. change is in the direction of $\vec{\nabla} f\big|_P$.

$\vec{\nabla} f = \langle f_x, f_y, f_z \rangle$

$= \langle 2x e^{-y}, x^2 e^{-y}(-1), 4 \rangle$

$\vec{\nabla} f\big|_P = \langle 12 e^{-\ln 2}, -36 e^{-\ln 2}, 4 \rangle$

$\boxed{e^{-\ln 2} = e^{\ln 2^{-1}} = 2^{-1} = \tfrac{1}{2}}$

$= \langle (\tfrac{1}{2})(12), -(\tfrac{1}{2})(36), 4 \rangle$

$= \langle 6, -18, 4 \rangle$

which has the same direction as $\langle 3, -9, 2 \rangle$.