1. The area of the surface obtained by rotating the curve

$$x = \cos^3(t),\ y = \sin^3(t),\ 0 \le t \le \frac{\pi}{2}$$

about the x-axis is

$\frac{dx}{dt} = 3\cos^2(t)\,(-\sin t)$

$\frac{dy}{dt} = 3\sin^2(t)\cdot(\cos t)$

(a) $\frac{6\pi}{5}$ ______ (correct)

(b) $\frac{4\pi}{5}$

(c) $\frac{3\pi}{5}$

(d) $\frac{2\pi}{5}$

(e) $\frac{\pi}{5}$

$S = 2\pi \int_\alpha^\beta y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$= 2\pi \int_0^{\pi/2} \sin^3(t) \sqrt{9\cos^4(t)\sin^2(t) + 9\sin^4(t)\cdot\cos^2(t)}\, dt$

$= 2\pi(3) \int_0^{\pi/2} \sin^3(t)\cdot |\cos(t)||\sin(t)| \underbrace{\sqrt{\cos^2(t)+\sin^2(t)}}_{1}\, dt$

$= 6\pi \int_0^{\pi/2} \sin^4(t)\cdot \underbrace{\cos(t)\, dt}_{dy}$

$= 6\pi \left.\frac{\sin^5(t)}{5}\right|_0^{\pi/2} = 6\pi\left(\frac{1}{5} - 0\right) = \frac{6\pi}{5}$

2. The polar curve

$$r = 1 + 2\cos(\theta)$$

is a

$\boxed{r = a + b\cos(\theta) \quad |b| > |a| \quad \text{limacon with inner loop}}$

(a) limacon with an inner loop ______ (correct)

(b) circle

(c) cardioid

(d) rose with 3 leaves

(e) rose with 6 leaves

[Details]

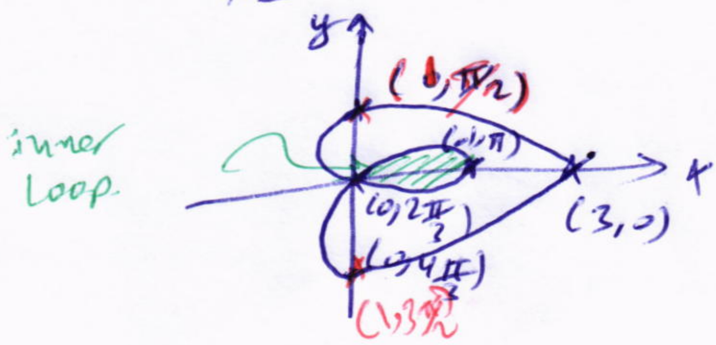
in $[0, 2\pi]$

$r = 0 \iff \theta = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$.

$\frac{2\pi}{3} \le \theta \le \pi$ | $\pi \le \theta \le \frac{4\pi}{3}$

$-1 \le \cos\theta \le \frac{-1}{2}$ | $-1 \le \cos\theta \le \frac{-1}{2}$

$-1 \le r \le 0$ | $-1 \le r \le 0$



3. The number of vectors $\vec{v}$ such that

$$\vec{v}\times \underbrace{<1,-1,3>}_{\vec{u}} = <1,1,2>$$

is

$\vec{u} \perp (\vec{v} \times \vec{u})$

So we should have

$\vec{u} \cdot (\vec{v} \times \vec{u}) = 0.$

(a) 0 ——————————————————(correct)

(b) 1

(c) 2

(d) 3

(e) ∞

However, $\vec{u} \cdot <1,1,2>$

$= <1,-1,3> \cdot <1,1,2>$

$= 1 - 1 + 6 = 6 \neq 0.$

$\therefore <1,1,2>$ cannot be $\vec{v} \times \vec{u}$ for any vector $\vec{v}$.

4. If θ is the angle between the planes

$$x + 2y + z = 1 \text{ and } 2x - y + z = 3$$

then $\cos(\theta) =$

$\vec{n_1} = <1, 2, 1>$

$\vec{n_2} = <2, -1, 1>$

(a) $\frac{1}{6}$ ——————————————————(correct)

(b) 0

(c) $-\frac{1}{6}$

(d) $-\frac{1}{3}$

(e) $\frac{1}{3}$

$$\cos(\theta) = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|}$$

$$= \frac{\cancel{2} \cancel{-2} + 1}{\sqrt{1+4+1}\sqrt{4+1+1}}$$

$$= \frac{1}{\sqrt{6}\sqrt{6}} = \frac{1}{6}$$

5. The intersection between the paraboloid $z = x^2 + y^2$ and the plane $2x + 2y - z = 2$ consists of

(a) one point
(b) a circle with radius 1
(c) a straight line
(d) two points
(e) a parabola

$z = x^2 + y^2 = 2x + 2y - 2$

$x^2 - 2x + y^2 - 2y = -2.$

$(x^2 - 2x + 1) + (y^2 - 2y + 1) = -2 + 2$

$(x-1)^2 + (y-1)^2 = 0.$

$(x, y) = (1, 1).$

6. The function

$$f(x, y) = x^2 + xy + y^2 + 8y$$

has at the **critical point** $P\left(\frac{8}{3}, -\frac{16}{3}\right)$

(a) a local minimum
(b) a local maximum
(c) a saddle point
(d) an absolute maximum
(e) a zero discriminant (whence the 2nd derivative test fails)

$f_x = 2x + y$

$f_y = x + 2y + 8.$

Notice that $f_x|_P = 0 = f_y|_P$

$f_{xx} = 2$
$f_{yy} = 2$
$f_{xy} = 1$

$D(x,y) = f_{xx} f_{yy} - f_{xy}^2$
$= (2)(2) - (1)^2$
$= 3 > 0.$

$f_{xx}|_P = 2 > 0.$

So $f(\frac{8}{3}, \frac{6}{13})$ is local min.

7. The function

$$f(x,y) = xy^2 - x^2y + 25x - 25y$$

has

(a) two saddle points
(b) one saddle point
(c) no saddle points
(d) three critical points
(e) infinite number of critical points

$f_x = -2xy + y^2 + 25$

$f_y = 2xy - x^2 - 25$

$\left.\begin{array}{l} f_x = 0 \\ f_y = 0 \end{array}\right\} \Rightarrow f_x + f_y = 0.$
$y^2 = x^2 ;\ y = \mp x.$

$\boxed{\text{Case I}}$ $y = x.$

$\begin{array}{l} f_x = 0 \\ f_y = 0 \end{array} \Rightarrow x = \mp 5.$
$(x,y) = (5,5)$ or $(-5,-5)$

$\boxed{\text{Case II}}$ $y = -x$

$f_x = 0 \Rightarrow 3x^2 + 25 = 0 \to \leftarrow$
Contradiction.

$\therefore \exists$ only two critical points

$f_{xx} = -2y,\ f_{yy} = 2x,\ f_{xy} = 2y - 2x.$

$D(x,y) = (-2y)(2x) - (2y - 2x)^2.$

$\boxed{x = y} \rightsquigarrow = -4x^2 < 0$
at both critical points.

8. The volume of the largest rectangular box in the **first octant** with three faces in the coordinate planes and one vertex in the plane

$$x + 2y + 3z = 6$$

is

(a) $\frac{4}{3}$
(b) $\frac{2}{3}$
(c) $\frac{5}{3}$
(d) $\frac{7}{3}$
(e) $\frac{8}{3}$

x : length
y : width $\Big\}\ > 0.$
z : height

$V = xyz$

$= \dfrac{xy(6 - x - 2y)}{3}$

$= \frac{1}{3}(6xy - x^2y - 2xy^2).$

$V_x = \frac{1}{3}(6y - 2xy - \cancel{2y^2}) = \frac{y}{3}(6 - 2x - 2y)$

$V_y = \frac{1}{3}(6x - \cancel{2x^2} - 4xy) = \frac{x}{3}(6 - x - 4y)$

$\boxed{V = \frac{4}{3}}$

$\begin{array}{l} V_x = 0 \\ V_y = 0 \end{array} \Rightarrow \left.\begin{array}{l} x + y = 3 \\ x + 4y = 6 \end{array}\right\}$

So, $3y = 3 ;\ y = 1$
$x = 2.$ $z = \frac{2}{3}.$

Only one critical point

$V_{xx} = \frac{1}{3}(-2y);$
$V_{yy} = \frac{1}{3}(-4x);$

$V_{xy} = \frac{1}{3}(6 - 2x - 4y)$

$V_{xx}(2,1) = -\frac{2}{3} < 0.$
$V(2,1)$ local max.

$D(2,1) = \left(-\frac{2}{3}\right)\left(-\frac{8}{3}\right) - \left(-\frac{2}{3}\right)^2 = \frac{12}{9} > 0$

absmax

9. If m is the minimum value and M is the maximum value of the function

$$f(x, y) = x - 3y - 1$$

subject to the constraint $\underbrace{x^2 + 3y^2}_{g(x,y)} = 16$, then $m + M =$

(a) -2
(b) -1
(c) 6
(d) 8
(e) 3

$\vec{\nabla f} = \lambda \vec{\nabla g}$

$1 = 2\lambda x$ — ①
$-3 = 6\lambda y$ — ②
$x^2 + 3y^2 = 16$ — ③

Clearly $x \neq 0$, $y \neq 0$ / $\lambda \neq 0$.

$\lambda = \frac{1}{2x} = \frac{-1}{2y}$

$\therefore y = -x$.

$x^2 + 3x^2 = 16$.
$x^2 = 4$.
$x = 2, y = -2$
or $x = -2, y = 2$.

$f(2, -2) = 7$ max.
$f(-2, 2) = -9$ min.
$m + M = -9 + 7 = -2$.

10. The minimum value m and the maximum value M of the function

$$f(x, y, z) = xz + y^2$$

on the intersection of the sphere $\underbrace{x^2 + y^2 + z^2}_{g(x,y,z)} = 4$ with the plane $z = x$ are

$z - x = h(x,y,z) = 0$

(a) $m = 2$ and $M = 4$
(b) $m = 3$ and $M = 5$
(c) $m = 3$ and $M = 4$
(d) $m = 2$ and $M = 3$
(e) $m = 0$ and $M = 1$

$\vec{\nabla f} = \lambda \vec{\nabla g} + \mu \vec{\nabla h}$

$z = 2\lambda x - \mu$ — ①
$\boxed{2y = 2\lambda y}$ — ②
$x = 2\lambda z + \mu$ — ③
$x^2 + y^2 + z^2 = 4$ — ④
$z = x$ — ⑤

From ②: $y = 0$ or $\lambda = 1$

Case I: $y = 0$

From ⑤ & ④

$2x^2 = 4$
$x^2 = 2$.

$(x, y, z) = (\sqrt{2}, 0, +\sqrt{2})$
or $(-\sqrt{2}, 0, -\sqrt{2})$

At these points:
$f(\sqrt{2}, 0, \sqrt{2}) = 2$
$= f(-\sqrt{2}, 0, -\sqrt{2})$

Case II: $y \neq 0$, $\lambda = 1$.

From ① & ③ & ⑤

$\cancel{x} = 2x - \mu$
$x = 2x + \mu$

$2x = 4x$; so $x = 0 = z$.

From ④: $y^2 = 4$; $y = \mp 2$.

$f(0, 2, 0) = 4 = f(0, -2, 0)$.

$\therefore m = 2$ & $M = 4$.

11. The volume of the solid under surface $z = xy$ and above the region R in the xy-plane bounded $y = x^2$ and $y = x$ is

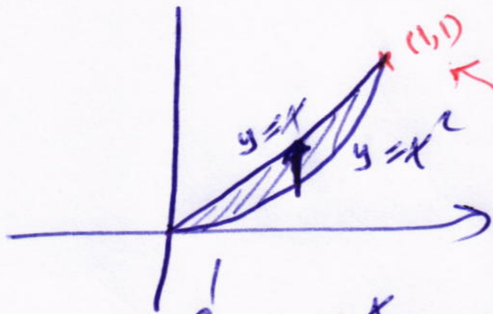
(a) $\frac{1}{24}$ ______________________________ (correct)

(b) $\frac{1}{12}$

(c) $\frac{1}{10}$

(d) $\frac{1}{8}$

(e) $\frac{1}{6}$



$x = x^2$
$x(1-x) = 0$
$x = 0$ or
$x = 1$

$$V = \int_0^1 \int_{x^2}^{x} (xy)\,dy\,dx = \int_0^1 \left(x \cdot \frac{y^2}{2}\Big|_{x^2}^{x}\right) dx$$
$$= \frac{1}{2}\int_0^1 (x^3 - x^5)\,dx = \frac{1}{2}\left(\frac{x^4}{4} - \frac{x^6}{6}\right)\Big|_0^1$$
$$= \frac{1}{2}\left(\frac{1}{4} - \frac{1}{6}\right) = \frac{1}{2}\left(\frac{3-2}{12}\right) = \frac{1}{24}$$

12. An estimate of the volume of the solid that lies below the surface $z = xy$ and above the rectangle

$$R = \{(x, y) |\ 0 \le x \le 6,\ 0 \le y \le 4\},$$

using a Riemann sum with $m = 3$, $n = 2$, and taking the sample point to be the **upper right corner** of each square is

(a) 288 ______________________________ (correct)

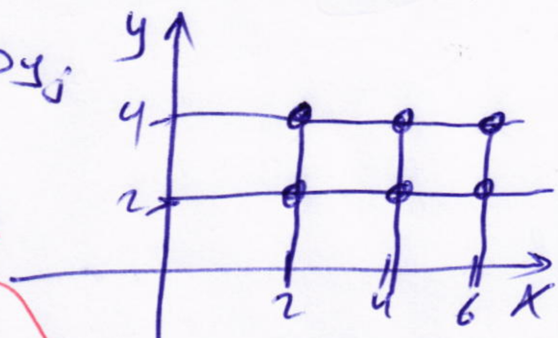
(b) 144

(c) 72

(d) 36

(e) 18

$$V \approx \sum_{j=1}^{2}\sum_{i=1}^{3} f(x_i^*, y_j^*)\,\Delta x_i\,\Delta y_j$$
$$= \sum_{j=1}^{2}\sum_{i=1}^{3} x_i^* y_j^* (2)(2)$$



$\boxed{\Delta x = \Delta y = 2}$

$$= 4 \sum_{j=1}^{2}\sum_{i=1}^{3} x_i^* y_j^*$$
$$= 4\left((2)(2) + 2(4) + 4(2) + 4(4) + 6(2) + 6(4)\right)$$
$$= 4(4 + 8 + 8 + 16 + 12 + 24)$$
$$= (4)(72) = 288.$$

13. The average value of the function $f(x,y) = x^2y$ on the rectangle $R : [-1,1] \times [0,5]$ is

(a) $\frac{5}{6}$

(b) $\frac{25}{3}$

(c) $\frac{5}{3}$

(d) $\frac{1}{10}$

(e) $\frac{2}{3}$

$$I = \int_0^5 \int_{-1}^1 x^2 y \, dx\, dy = \int_0^5 \left(\frac{x^3 y}{3} \Big|_{-1}^{1} \right) dy$$

$$= \int_0^5 \frac{2y}{3} \, dy$$

$$= \frac{y^2}{3} \Big|_0^5 = \frac{25}{3}$$

$$\boxed{\begin{aligned} &\text{Area}(R) = \\ &(1-(-1))\,(5-0) \\ &= (2)(5) = 10 \end{aligned}}$$

$$\text{Average}_R(f) = \frac{I}{\text{Area}(R)} = \frac{\frac{25}{3}}{10} = \frac{25}{30} = \frac{5}{6}$$

14. $\int_0^2 \int_x^2 e^{-y^2} \, dy\, dx =$

(a) $\frac{e^4 - 1}{2e^4}$

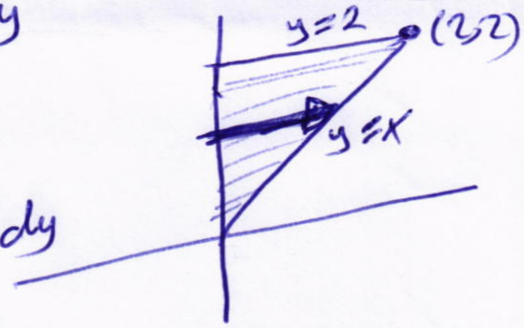
(b) $\frac{e^4 + 1}{2e^4}$

(c) $\frac{1 - e^4}{2e^4}$

(d) $\frac{e^4 - 1}{e^4}$

(e) $\frac{e^4 + 1}{e^4}$

$$\int_0^2 \int_0^y e^{-y^2} \, dx\, dy$$



$$= \int_0^2 \left(e^{-y^2} x \Big|_0^y \right) dy$$

$$= \int_0^2 \left(y e^{-y^2} - 0 \right) dy$$

$$= -\frac{1}{2} \int_0^2 \left(-2y e^{-y^2} \right) dy$$

$$= -\frac{1}{2} e^{-y^2} \Big|_0^2$$

$$= -\frac{1}{2} \left(e^{-4} - 1 \right) = -\frac{1}{2} \left(\frac{1}{e^4} - 1 \right)$$

$$= -\left(\frac{1 - e^4}{2e^4} \right)$$

$$= \frac{e^4 - 1}{2e^4}$$

$z = 2 - y$

15. The volume of the solid bounded by the cylinder $x^2+y^2=9$ and the planes $\boxed{y+z=2}$ and $z=0$ is

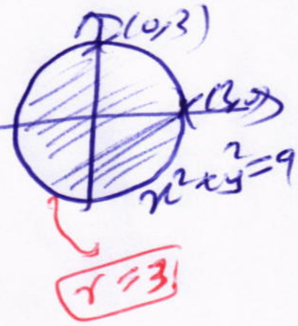
$\boxed{z = 2-y}$

(a) 18π
(b) 20π
(c) 24π
(d) 10π
(e) 9π

$$V = \iint_R ((2-y)-0)\,dA$$

Jacobian

$$= \int_0^{2\pi}\int_0^3 (2 - r\sin\theta)\,\boxed{r}\,dr\,d\theta$$



$\boxed{r=3}$

$$= \int_0^{2\pi}\int_0^3 (2r - r^2\sin\theta)\,dr\,d\theta$$

$$= \int_0^{2\pi} \left(r^2 - \frac{r^3}{3}\sin\theta\right)\Big|_0^3\,d\theta$$

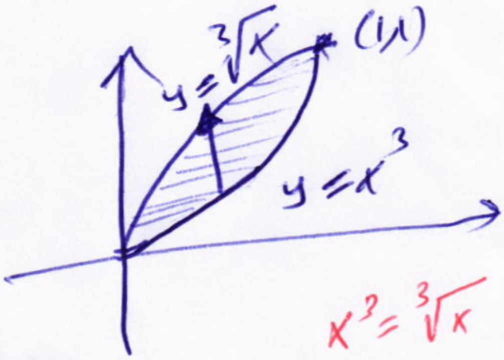
$$= \int_0^{2\pi} (9 - 9\sin\theta)\,d\theta = (9\theta - 9\cos\theta)\Big|_0^{2\pi} = 18\pi.$$

16. If D is the region in the **first quadrant** bounded by $y=\sqrt[3]{x}$ and $y=x^3$, then

$$99\iint_D xy^2\,dA = 99\int_0^1\int_{x^3}^{\sqrt[3]{x}} xy^2\,dy\,dx$$

(a) 8
(b) 6
(c) 4
(d) 2
(e) 1

$$= 99\int_0^1 \left(\frac{x\,y^3}{3}\Big|_{x^3}^{\sqrt[3]{x}}\right)dx$$



$x^3 = \sqrt[3]{x}$
$x^9 = x$
$x(x^8-1)=0$
$x(x-1)(x+1)(x^2+1)(x^4+1)=0$
$x = 0,\ 1$

$$= \frac{99}{3}\int_0^1 (x^2 - x^{10})\,dx$$

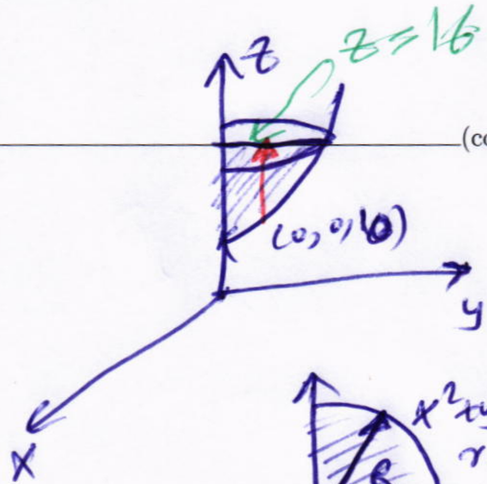
$$= 33\left(\frac{x^3}{3} - \frac{x^{11}}{11}\right)\Big|_0^1$$

$$= 33\left(\left(\frac{1}{3} - \frac{1}{11}\right) - 0\right)$$

$$= 33\,\frac{11-3}{33} = 8.$$

$\boxed{x=-1 \text{ rejected}}$

17. The volume of the solid in the **first octant** bounded by the paraboloid $z = 2x^2 + 2y^2 + 10$ and the plane $z = 16$ is given by



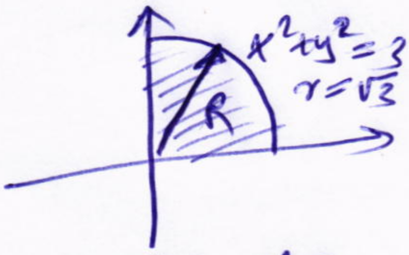
(a) $\int_0^{\frac{\pi}{2}} \int_0^{\sqrt{3}} (6r - 2r^3)\, dr\, d\theta$ ________________ (correct)

(b) $\int_0^{2\pi} \int_0^{\sqrt{3}} (6r - 2r^3)\, dr\, d\theta$

(c) $\int_0^{\pi} \int_0^{3} (6r - 2r^3)\, dr\, d\theta$

(d) $\int_0^{\frac{\pi}{2}} \int_0^{\sqrt{3}} (3r - 2r^3)\, dr\, d\theta$

(e) $\int_0^{\frac{\pi}{2}} \int_0^{\sqrt{3}} (3r - 2r^2)\, dr\, d\theta$



$V = \iint_R \left(16 - (2(x^2+y^2)+10)\right) dz\, dA$

first quadrant

$= \int_0^{\pi/2}\int_0^{\sqrt{3}} (6 - 2r^2)\,\boxed{r}\, dr\, d\theta.$

Jacobian

18. $\int_{-1}^{1} \int_0^{\sqrt{1-y^2}} (x^2 + y^2)^2\, dx\, dy =$

$0 \le x \le \sqrt{1-y^2}$, $-1 \le y \le 1$

$x^2 + y^2 \le 1$

$= \int_{-\pi/2}^{\pi/2} \int_0^1 (r^2)^2\,\boxed{r}\, dr\, d\theta$

Jacobian

(a) $\frac{\pi}{6}$ ________________ (correct)

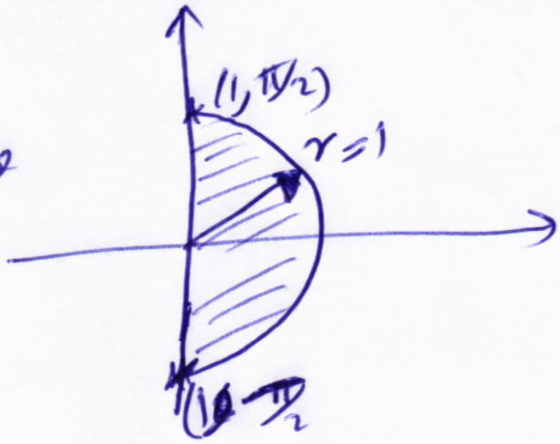
(b) $\frac{\pi}{5}$

(c) $\frac{\pi}{4}$

(d) $\frac{\pi}{3}$

(e) $\frac{\pi}{2}$

$= \int_{-\pi/2}^{\pi/2} \int_0^1 r^5\, dr\, d\theta$



$= \int_{-\pi/2}^{\pi/2} \frac{r^6}{6}\Big|_0^1 d\theta$

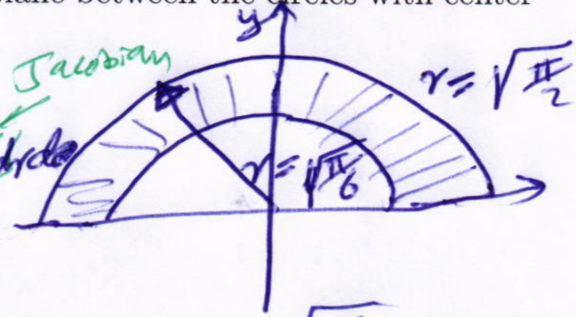
$= \frac{1}{6} \int_{-\pi/2}^{\pi/2} d\theta = \frac{1}{6}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right)$

$= \frac{\pi}{6}.$

19. If R is the region in the **upper half** of the xy-plane between the circles with center the origin and radii $\sqrt{\frac{\pi}{6}}$ and $\sqrt{\frac{\pi}{2}}$, then

Jacobian

$$\int\int_R \cos(x^2+y^2)\,dA = \int_0^{\pi}\int_{\sqrt{\frac{\pi}{6}}}^{\sqrt{\frac{\pi}{2}}} \cos(r^2)\,\boxed{r\,dr\,d\theta}$$



(a) $\frac{\pi}{4}$ ——— (correct)
(b) $\frac{\pi}{8}$
(c) $\frac{\pi}{16}$
(d) $\frac{\pi}{2}$
(e) 0

$$= \int_0^{\pi}\left(\frac{\sin(r^2)}{2}\Bigg|_{\sqrt{\frac{\pi}{6}}}^{\sqrt{\frac{\pi}{2}}}\right)d\theta$$

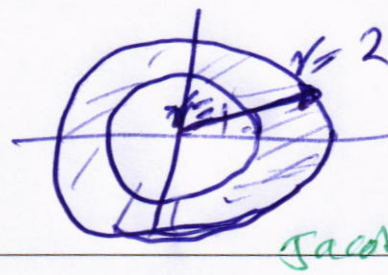
$$= \frac{1}{2}\int_0^{\pi}\left(\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{6}\right)\right)d\theta$$

$$= \frac{1}{2}\int_0^{\pi}\left(1-\frac{1}{2}\right)d\theta = \frac{1}{4}\,\theta\Big|_0^{\pi} = \frac{\pi - 0}{4} = \frac{\pi}{4}$$

20. The volume of the solid under the cone $z = \sqrt{x^2+y^2}$ and above the ring

$$R = \{(x,y)|\ 1 \le x^2+y^2 \le 4\}$$

is



(a) $\frac{14\pi}{3}$ ——— Jacobian (correct)
(b) 4π
(c) $\frac{7\pi}{3}$
(d) 3π
(e) 2π

$$V = \int_0^{2\pi}\int_1^2 (r - 0)\,\boxed{r}\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^2 r^2\,dr\,d\theta$$

$$= \int_0^{2\pi}\left(\frac{r^3}{3}\Bigg|_1^2\right)d\theta$$

$$= \int_0^{2\pi}\frac{8-1}{3}\,d\theta = \frac{7}{3}\left(\theta\Big|_0^{2\pi}\right) = \frac{14\pi}{3}$$

21. $\int_1^2 \int_z^2 \int_0^{\sqrt{3}y} \frac{y}{x^2+y^2}\,dx\,dy\,dz =$ $\int_1^2\int_z^2 \left(\left(\frac{1}{\cancel{y}}\cdot \tan^{-1}\left(\frac{x}{y}\right)\right)\cancel{y}\right)\Big|_0^{\sqrt{3}y} dy\,dz$

(a) $\frac{\pi}{6}$
(b) $\frac{2}{\pi}$
(c) 3π
(d) π
(e) $\frac{\pi}{8}$

$= \int_1^2\int_z^2 \left(\tan^{-1}(\sqrt{3}) - \tan^{-1}(0)\right) dy\,dz$

$= \int_1^2\int_z^2 \frac{\pi}{3}\, dy\,dz$

$= \frac{\pi}{3}\int_1^2 \left(y\Big|_z^2\right) dz = \frac{\pi}{3}\int_1^2 (2-z)\,dz$

$= \frac{\pi}{3}\left(2z - \frac{z^2}{2}\right)\Big|_1^2$

$= \frac{\pi}{3}\left((4-2) - (2-\frac{1}{2})\right)$

$= \frac{\pi}{3}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

22. $\int_0^1 \int_0^{1-x} \int_{x+z}^1 dy\,dz\,dx =$

(a) $\frac{1}{6}$
(b) $\frac{1}{5}$
(c) $\frac{1}{4}$
(d) $\frac{1}{3}$
(e) $\frac{1}{2}$

$\int_0^1\int_0^{1-x} y\Big|_{x+z}^1 dz\,dx$

$= \int_0^1\int_0^{1-x} (1-x-z)\,dz\,dx$

$= \int_0^1 \left(z - xz - \frac{z^2}{2}\right)\Big|_0^{1-x} dx = \int_0^1 \left((1-x) - x(1-x) - \frac{(1-x)^2}{2}\right) dx$

$= \int_0^1 (1-x)\left((1-x) - \frac{(1-x)}{2}\right) dx$

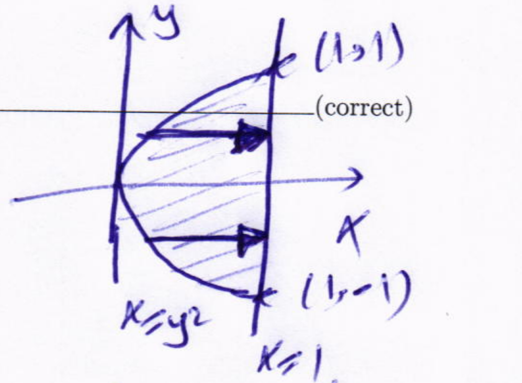
$= \frac{1}{2}\int_0^1 (1-x)^2\,dx = \frac{1}{2}\int_0^1 (1-2x+x^2)\,dx$

$= \frac{1}{2}\left(x - x^2 + \frac{x^3}{3}\right)\Big|_0^1$

$= \frac{1}{2}\left(1-1+\frac{1}{3}\right) = \frac{1}{6}$

23. The volume of the solid bounded by the planes $z = 0$, $z = x$, $x = 1$ and the parabolic cylinder $x = y^2$ is

$$V = \int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{x} dz\,dx\,dy$$



(a) $\frac{4}{5}$ ——— (correct)

(b) $\frac{3}{5}$

(c) $\frac{2}{5}$

(d) $\frac{1}{5}$

(e) $\frac{7}{5}$

$$= \int_{-1}^{1}\int_{y^2}^{1} z\Big|_{0}^{x}\,dx\,dy$$

$$= \int_{-1}^{1}\int_{y^2}^{1} x\,dx\,dy = \int_{-1}^{1}\left(\frac{x^2}{2}\Big|_{y^2}^{1}\right)dy = \int_{-1}^{1}\frac{1-y^4}{2}\,dy$$

$$= \frac{1}{2}\left(y-\frac{y^5}{5}\right)\Big|_{-1}^{1} = \frac{1}{2}\left(\left(1-\frac{1}{5}\right)-\left(-1+\frac{1}{5}\right)\right)$$

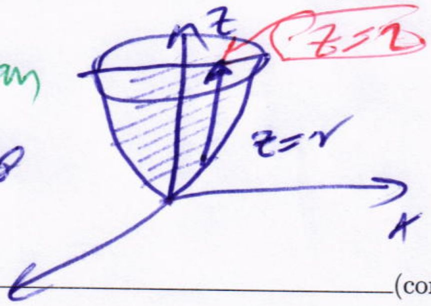
$$= \frac{1}{2}\left(\frac{8}{5}\right) = \frac{4}{5}.$$

24. If E is the solid enclosed by the cone $z = \sqrt{x^2+y^2}$ and the plane $z = 2$, then

$$I = \iiint_E (x^2+y^2)\,dV =$$

Jacobian

$$I = \int_{0}^{2\pi}\int_{0}^{2}\int_{r}^{2} r^2\,\boxed{r}\,dz\,dr\,d\theta$$



(a) $\frac{16\pi}{5}$ ——— (correct)

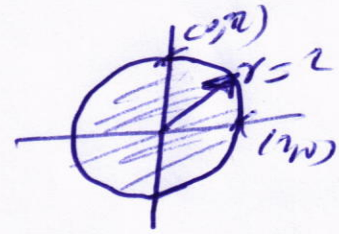
(b) $\frac{8\pi}{5}$

(c) $\frac{32\pi}{5}$

(d) 4π

(e) π

$$= \int_{0}^{2\pi}\int_{0}^{2}\int_{r}^{2} r^3\,dz\,dr\,d\theta$$



$$= \int_{0}^{2\pi}\int_{0}^{2} r^3(2-r)\,dr\,d\theta$$

$$= \int_{0}^{2\pi}\int_{0}^{2} (2r^3-r^4)\,dr\,d\theta$$

$z=2$

$\sqrt{x^2+y^2}=2$

$x^2+y^2=4.$

$$= \int_{0}^{2\pi}\left(2\cdot\frac{r^4}{4}-\frac{r^5}{5}\right)\Big|_{0}^{2}\,d\theta$$

$$= \int_{0}^{2\pi}\left(8-\frac{32}{5}\right)d\theta = \frac{8}{5}\int_{0}^{2\pi}d\theta = \frac{8}{5}\left(\theta\Big|_{0}^{2\pi}\right)$$

$$= \frac{8}{5}(2\pi-0) = \frac{16\pi}{5}$$

25. The volume of the solid in the **first octant** bounded above by the paraboloid $z = 4 - x^2 - y^2$ and on the sides by the cylinder $\boxed{x^2 + y^2 = 2x}$ is $\to (x-1)^2 + y^2 = 1$

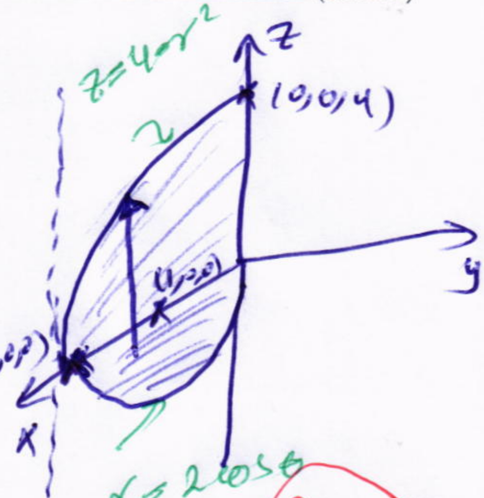
Jacobian $\quad r^2 = 2r\cos\theta \quad r = 2\cos\theta$

$$V = \int_0^{\pi/2}\int_0^{2\cos\theta} (4 - r^2)\,\boxed{r}\,dr\,d\theta$$

(a) $\frac{5\pi}{4}$ (correct)
(b) $\frac{3\pi}{4}$
(c) $\frac{\pi}{3}$
(d) $\frac{\pi}{4}$
(e) $\frac{5\pi}{2}$

$$= \int_0^{\pi/2}(4r - r^3)\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left(2r^2 - \frac{r^4}{4}\right)\Big|_0^{2\cos\theta} d\theta$$



$$= \int_0^{\pi/2}(8\cos^2\theta - 4\cos^4\theta)\,d\theta$$

$$= 4\int_0^{\pi/2}\cos^2\theta\,(2 - \cos^2\theta)\,d\theta = 4\int_0^{\pi/2}\cos^2\theta\,(1 + \sin^2\theta)\,d\theta$$

$$\sin^2(2\theta) = 1 - \cos^2(2\theta)$$

$$= \cancel{4}\,2\int_0^{\pi/2}\frac{1 + \cos(2\theta)}{\cancel{2}}\,d\theta + \cancel{4}\int_0^{\pi/2}\frac{1 + \cos(2\theta)}{\cancel{2}}\cdot\frac{1 - \cos(2\theta)}{\cancel{2}}\,d\theta$$

26. The surface whose equation in spherical coordinates is given by $\rho = 5\csc(\phi)\csc(\theta)$ is a

$$= 2\left(\theta + \frac{\sin 2\theta}{2}\right)\Big|_0^{\pi/2} + \int_0^{\pi/2}\frac{1 - \cos(4\theta)}{2}\,d\theta$$

(a) plane (correct)
(b) sphere
(c) cone
(d) cylinder
(e) paraboloid

$$= 2\left(\frac{\pi}{2} + 0\right) + \frac{1}{2}\left(\theta - \frac{\sin(4\theta)}{4}\right)\Big|_0^{\pi/2}$$

$$= \pi + \frac{\pi}{4} = \frac{5\pi}{4}.$$

$$\rho = 5\csc(\phi)\csc(\theta)$$

$$\rho = 5\,\frac{1}{\sin\phi}\cdot\frac{1}{\sin\theta}$$

$$\boxed{\rho\sin\phi}\,\sin\theta = 5$$

$$r\sin\theta = 5$$

$$\boxed{y = 5}$$

plane parallel to the xz-plane.

27. The volume of the part of ball $x^2 + y^2 + z^2 \leq 9$ that lies between the upper and the lower halves of the cone

$$3z^2 = x^2 + y^2$$

is

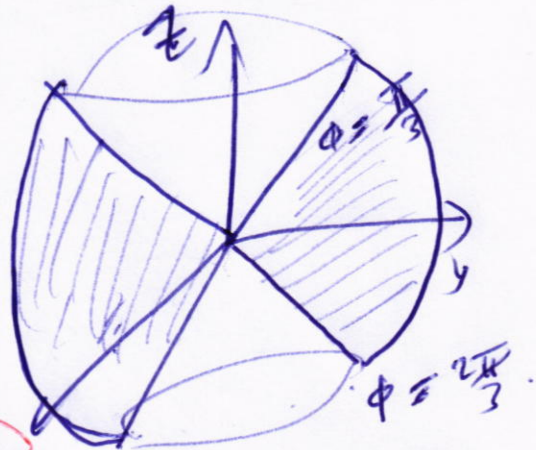
$\rho \leq 3$

$3z^2 = r^2$

$\frac{x^2}{z^2} = 3$

$\tan\phi = \frac{x}{z} = \mp\sqrt{3}$

$\phi = \frac{\pi}{3}$ $\boxed{\text{or}}$ $\phi = \frac{2\pi}{3}$



(a) 18π
(b) 15π
(c) 12π
(d) 9π
(e) 6π

Jacobian

$$V = \int_0^{2\pi}\int_{\pi/3}^{2\pi/3}\int_0^3 \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_{\pi/3}^{2\pi/3} \left(\frac{\rho^3}{3}\Big|_0^3 \sin\phi\right) d\phi\, d\theta$$

$$= 9\int_0^{2\pi}\int_{\pi/3}^{2\pi/3} \sin\phi \, d\phi\, d\theta = 9\int_0^{2\pi} \left(-\cos\phi\Big|_{\pi/3}^{2\pi/3}\right) d\theta = 9\int_0^{2\pi} -\left(-\frac{1}{2} - \frac{1}{2}\right) d\theta$$

$$9(2\pi) = 18\pi.$$

28. If E is the region that lies between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$, then

$$\iiint_E (x^2 + y^2)\, dV = \iiint_E r^2 \cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

Jacobian

$$= \int_0^{2\pi}\int_0^{\pi}\int_1^2 \rho^4 \cdot \sin^3\phi \, d\rho\, d\phi\, d\theta$$

$r = \rho \sin\phi$
$r^2 = \rho^2 \sin^2\phi$

(a) $\frac{248\pi}{15}$
(b) $\frac{124\pi}{15}$
(c) $\frac{62\pi}{15}$
(d) $\frac{31\pi}{15}$
(e) $\frac{16\pi}{15}$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{\rho^5}{5}\Big|_1^2 \cdot \sin\phi\,(1 - \cos^2\phi)\, d\phi\, d\theta$$

$$= \frac{31}{5}\int_0^{2\pi}\int_0^{\pi} (\sin\phi - \sin\phi\cos^2\phi)\Big|_0^{\pi} d\phi\, d\theta$$

$$= \frac{31}{5}\int_0^{2\pi} \left(-\cos\phi + \frac{\cos^3\phi}{3}\right)\Big|_0^{\pi} d\theta = \frac{31}{5}\left(2 - \frac{2}{3}\right)\cdot(2\pi)$$

$$\frac{31}{5}\left(\frac{4}{3}\right)(2\pi) = \frac{248\pi}{15}$$

Answer KEY

Q	MASTER	CODE01	CODE02	CODE03	CODE04
1	A	C 2	C 3	E 3	C 4
2	A	A 1	D 2	E 1	C 1
3	A	C 3	B 1	D 2	A 2
4	A	E 4	B 4	B 4	C 3
5	A	D 8	B 8	E 5	D 8
6	A	A 7	B 7	C 7	A 6
7	A	B 5	B 5	C 8	A 7
8	A	C 6	A 6	E 6	A 5
9	A	C 11	B 12	D 9	A 11
10	A	E 9	A 11	D 10	B 9
11	A	D 12	E 10	C 12	D 12
12	A	A 10	E 9	D 11	E 10
13	A	E 16	A 14	D 15	C 16
14	A	D 15	E 15	B 14	D 14
15	A	C 14	E 13	C 13	D 13
16	A	B 13	B 16	A 16	E 15
17	A	D 19	E 19	A 19	A 18
18	A	D 20	B 17	C 17	D 17
19	A	D 18	E 18	B 18	A 19
20	A	D 17	D 20	E 20	D 20
21	A	C 22	C 23	D 23	C 21
22	A	B 24	A 24	A 22	C 24
23	A	A 21	E 22	B 24	E 23
24	A	B 23	C 21	B 21	D 22
25	A	A 28	D 28	D 25	D 27
26	A	D 26	D 27	D 28	D 28
27	A	B 25	C 25	D 26	B 25
28	A	B 27	B 26	B 27	D 26