King Fahad University of Petroleum and Minerals Department of Mathematics

MATH 202-Exam 1- Term 211

Duration : 90 Minutes Code 001

Name: Key ID Number: ____________

Section Number: ____________ Serial Number: ____________

Class Time: ____________ Instructor`s Name: ____________

Instructions:

1. Calculators and Mobile Phones are not allowed.
2. Write Legibly. You may lose points for messy work.
3. Make sure that you have 8 pages of problems (Total of 11 Problems)
   (There are 7 multiple choice and 4 written questions).
4. For the written questions, you need to show all your work. No points for answer without justifications.

Question number	**Answer**	**Maximum Points**	**Points**
1	e	5	
2	b	5	
3	d	5	
4	d	5	
5	c	5	
6	b	5	
7	e	5	
Total		**35**	

Written

Question number	**Points**	**Maximum Points**
8		10
9		10
10		10
11		10
Total		**40**

75

1. The **sum** of all values of m such that $y = e^{mx}$ is a solution of the DE

$$y'' - 6y' - 7y = 0$$

is

let $y = e^{mx} \Rightarrow y' = m e^{mx} \Rightarrow y'' = m^2 e^{mx}$

Substituting in the DE, we get

$m^2 e^{mx} - 6m e^{mx} - 7 e^{mx} = 0 \Rightarrow$

$e^{mx} (m^2 - 6m - 7) = 0 \Rightarrow e^{mx} (m-7)(m+1) = 0$

$e^{mx} \neq 0$ so $(m-7)(m+1) = 0 \Rightarrow m = 7$ or $m = -1$

Sum $= 7 - 1 = 6$

a) 0
b) 4
c) 5
d) 2
$\boxed{\text{e)}}$ 6

2. The **sum** of all constant solutions of the DE

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y^2 = 4$$

is

let $y = K$ where K is constant

$y' = 0$, $y'' = 0$.

$\therefore$ $K^2 = 4$ or $K = \pm 2$.

Sum $= -2 + 2 = 0$

a) 4
$\boxed{\text{b)}}$ 0
c) 2
d) -2
e) 6

3. An integrating factor that makes the DE equation

$$\underbrace{y(x^3+y^3)}_{M(x,y)}\,dx + \underbrace{(x^4+7xy^3)}_{N(x,y)}\,dy = 0$$

exact is

a) y

b) y^4

c) y^2

$\boxed{\text{d)}}$ y^3

e) y^5

$$M_y = x^3 + 4y^3 \quad , \quad N_x = 4x^3 + 7y^3$$

$$\frac{N_x - M_y}{M} = \frac{4x^3+7y^3-x^3-4y^3}{y(x^3+y^3)} = \frac{3x^3+3y^3}{y(x^3+y^3)} = \frac{3}{y}\cdot\frac{\cancel{(x^3+y^3)}}{\cancel{(x^3+y^3)}}$$

$$\therefore \mu(y) = e^{\int \frac{3}{y}\,dy} = e^{3\ln y} = y^3 \,,\ y>0.$$

4. A thermometer reading $70°\,F$ is taken out where the temperature is $20°\,F$. Four minutes later, the reading is $30°\,F$. The thermometer's reading 8 minutes after it was brought outside is

a) $28°\,F$

b) $24°\,F$

c) $26°\,F$

$\boxed{\text{d)}}$ $22°\,F$

e) $25°\,F$

$$T - T_m = c\,e^{kt}$$

$$T(0) = 70 \quad , \quad T(4) = 30$$

$$T_m = 20 \qquad T(8) = ?$$

$$T(0) = 70 \Rightarrow 70 - 20 = C \Rightarrow C = 50$$

$$T(4) = 30 \Rightarrow 30 - 20 = 50\,e^{4k}$$

$$\Rightarrow e^{4k} = \frac{1}{5} \Rightarrow 4k = \ln\left(\frac{1}{5}\right) \Rightarrow k = \frac{-\ln 5}{4}$$

$$\therefore T(t) = 20 + 50\,e^{\left(\frac{-\ln 5}{4}\right)t}$$

$$\Rightarrow T(8) = 20 + 50\cdot e^{-2\ln 5} = 20 + 50\left(\frac{1}{25}\right)$$

$$= 20 + 2 = 22°\,F$$

5. Which one of the following is a $4th$ order, linear differential equation

a) $(y')^4 + xy = \sin x$

b) $y'''' + yy' + e^x y = 0$

$\boxed{\text{c)}}$ $x^4 y'''' + e^x y'' + y = \ln x$

d) $y'''' + xy''' + \sin(xy) = e^x$

e) $y'''' + x^2 y'' + xy' = \sin(y)$

6. According to the Existance and Uniqueness Theorem, the IVP $\dfrac{dy}{dx} = \dfrac{\sqrt{4-y^2}}{x}$, $y(\alpha) = \beta$ has a unique solution if

a) $\alpha = 0,\ \beta = 1$

$\boxed{\text{b)}}$ $\alpha = 1,\ \beta = -1$

c) $\alpha = 0,\ \beta = 0$

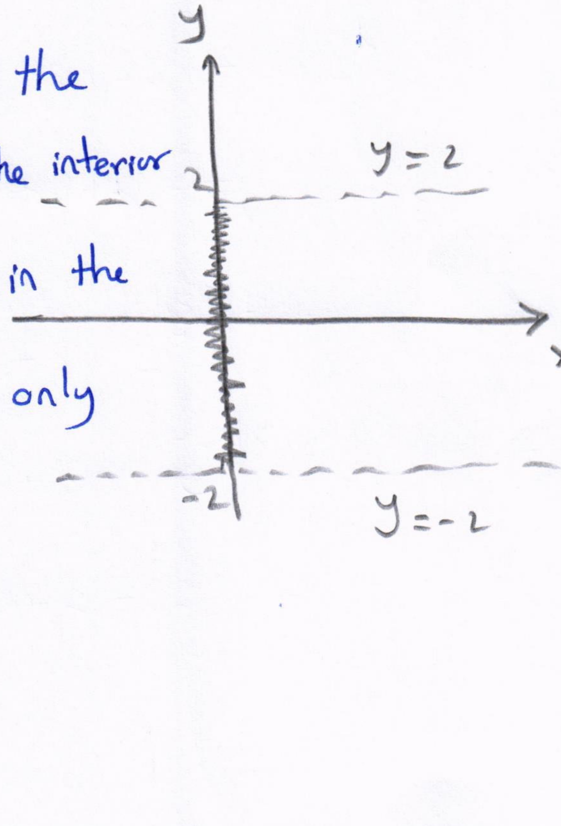
d) $\alpha = 1,\ \beta = 3$

e) $\alpha = 2,\ \beta = -4$

$$f(x,y) = \frac{\sqrt{4-y^2}}{x}$$

$$\frac{\partial f(x,y)}{\partial y} = \frac{-y}{x\sqrt{4-y^2}}$$

Since f and $\frac{\partial f}{\partial y}$ are both continuous for $x \neq 0$ and $y \in (-2, 2)$ then we need the point to be contained in the interior of a rectangle that is contained in the region of continuity and the only point is $(1, -1)$



7. The solution curve of the initial-value problem

$$x\frac{dy}{dx}+3y=6x^3,\ y(1)=1$$

passes through the point

a) $(4,2)$
b) $(3,9)$
c) $(2,-2)$
d) $(3,-8)$
$\boxed{\text{e)}}$ $(2,8)$

(1) $\frac{dy}{dx}+\frac{3}{x}=6x^2$. Linear in y.

$u(x)=e^{\int \frac{3}{x}dx}=e^{3\ln x}=e^{\ln x^3}=x^3,\ x>0.$

multiply the DE (1) by x^3. This gives;

$\frac{d}{dx}(x^3y)=6x^5 \Rightarrow x^3y=x^6+C$

$\Rightarrow y=x^3+Cx^{-3}$.

$y(1)=1 \Rightarrow 1=1+C \Rightarrow C=0$

$\therefore\ y(x)=x^3$. The only point satisfies the solution is $(2,8)$

8. **(10 points)** Find a family of solutions for the DE

$$x \tan y - (\sec x)\frac{dy}{dx} = 0$$

$$(x \tan y)\,dx - (\sec x)\,dy = 0$$

$$\Rightarrow \frac{x}{\sec x}\,dx - \cot y\,dy = 0 \qquad \text{Separable DE.}$$

$$\Rightarrow x\cos x\,dx - \cot y\,dy = 0$$

$$\Rightarrow \int x\cos x\,dx = \int \cot y\,dy \qquad (3\text{pts})$$

$$\Rightarrow \underbrace{x\sin x + \cos x}_{(3\text{pts})} = \underbrace{\ln|\sin y|}_{(3\text{pts})} + \underbrace{C}_{(1\text{pt})}$$

or

$$x\sin x + \cos x - \ln|\sin y| = C$$

9. **(10 points)** Find the general solution of the exact DE

$$(3y^2\,e^{3x} - y\sin(xy) + \ln x)\,dx - (x\,\sin(xy) - 2y\,e^{3x})\,dy = 0$$

$M(x,y) = 3y^2 e^{3x} - y\sin(xy) + \ln x$ (1 pt)

$N(x,y) = -x\sin(xy) + 2y e^{3x}$ (1 pt)

$M_y = N_x$ (no need to check).

Since the DE is exact $\Rightarrow$ there is a function $f(x,y)$ s.t

$\frac{\partial f}{\partial x} = M(x,y) = 3y^2 e^{3x} - y\sin(xy) + \ln x$ and

$\frac{\partial f}{\partial y} = N(x,y) = -x\sin(xy) + 2y e^{3x}$ (2 pts)

$\Rightarrow f(x,y) = \int \frac{\partial f}{\partial y}\,dy = \int(-x\sin(xy) + 2y e^{3x})\,dy$

$= \cos(xy) + y^2 e^{3x} + g(x)$. (2 pts)

Now, $\frac{\partial f}{\partial x} = M(x,y) \Rightarrow$

$\cancel{-y\sin(xy)} + \cancel{3y^2 e^{3x}} + g'(x) = \cancel{3y^2 e^{3x}} \cancel{- y\sin(xy)} + \ln x$

$\Rightarrow g'(x) = \ln x \Rightarrow g(x) = x\ln x - x + C_1$ (2 pts)

The solution of the D.E is $f(x,y) = C_2$ or

$\cos(xy) + y^2 e^{3x} + x\ln x - x = C$ (2 pts)

10. **(10 points)** Use a suitable substitution to solve the initial-value problem

$$x\frac{dy}{dx} - y = y^2 \ln x, \; y(1) = \frac{1}{2}$$

$$\frac{dy}{dx} - \frac{1}{x}y = \frac{\ln x}{x} y^2$$ Bernoulli DE

let $u = y^{1-2} = y^{-1}$

$\Rightarrow \frac{du}{dx} = -y^{-2}\frac{dy}{dx}$ (2pts)

$$\Rightarrow y^{-2}\frac{dy}{dx} - \frac{1}{x}y^{-1} = \frac{\ln x}{x}$$

$$\Rightarrow -\frac{du}{dx} - \frac{1}{x}u = \frac{\ln x}{x}$$

$$\Rightarrow \frac{du}{dx} + \frac{1}{x}u = -\frac{\ln x}{x} \quad \cdots (1)$$ (2pts)

Integrating factor $e^{\int \frac{1}{x}dx} = e^{\ln x} = x, \; x > 0.$ (2pts)

multiply the DE in (1) by x to get

$$\frac{d}{dx}(xu) = -\ln x$$

or

$$xu = -x\ln x + x + C$$ (2pts)

$\Rightarrow u = -\ln x + 1 + \frac{C}{x}$ or $\frac{1}{y} = -\ln x + 1 + \frac{C}{x}$ (1 pt)

$y(1) = \frac{1}{2} \Rightarrow 2 = 1 + C \Rightarrow C = +1$ (1pt)

and the solution of the IVP is $\frac{1}{y} = -\ln x + 1 + \frac{1}{x}$

11. **(10 points)** Solve the homogeneous DE

$$\frac{dy}{dx} = \frac{x}{y} + \frac{y}{x}.$$

$$xy\frac{dy}{dx} = x^2 + y^2 \Rightarrow xy\,dy = (x^2+y^2)\,dx$$

$$\Rightarrow (x^2+y^2)\,dx - xy\,dy = 0 \quad \cdots (1)$$

let $y = ux \Rightarrow dy = u\,dx + x\,du$ (3pts)

Substitute in (1):

$$(x^2+u^2x^2)\,dx - x(ux)(u\,dx + x\,du) = 0$$

$$\Rightarrow (1+u^2)\,dx - u(u\,dx + x\,du) = 0$$

$$\Rightarrow dx - ux\,du = 0 \Rightarrow \frac{dx}{x} = u\,du \quad \text{(3pts)}$$

$$\Rightarrow \ln|x| = \frac{u^2}{2} + C \quad \text{(2pts)}$$

$$\Rightarrow \ln|x| = \frac{1}{2}\left(\frac{y}{x}\right)^2 + C. \quad \text{(2pts)}$$