King Fahad University of Petroleum and Minerals Department of Mathematics

MATH 202-Exam II - Term 211

Duration : 90 Minutes Master

Name:______________________________ ID Number: __________________________

Section Number: ___________________ Serial Number: _______________________

Class Time: _________________________ Instructor`s Name: _____________________

Instructions:

1. Calculators and Mobile Phones are not allowed.
2. Write Legibly. You may lose points for messy work.
3. Make sure that you have 8 pages of problems (Total of 11 Problems)
   (There are 7 multiple choice and 4 written questions).
4. For the written questions, you need to show all your work. No points for answer without justifications.
5. For the MCQ, you must insert your answers in the table at the cover page.

Question number	Answer	Maximum Points	Points
1		5	
2		5	
3		5	
4		5	
5		5	
6		5	
7		5	
Total		35	

Written

Question number	Points	Maximum Points
8		8
9		10
10		10
11		12
Total		40

75

1. Let $y(x)$ be the solution of the boundary-value problem

$$x^2y'' + 3xy' - 3y = 0, \quad y(1) = 0, \quad y(2) = 15.$$

Then $y'(1) =$

a) 32
b) 30
c) 28
d) 26
e) 24

2. Using the substitution $x = e^t$ to transform the Cauchy-Euler differential equation

$$x^2y'' - 3xy' + 13y = 4 + 3x$$

to a differential equation with constant coefficients we get:

a) $y'' - 4y' + 13y = 4 + 3e^t$
b) $y'' + 4y' + 13y = 4 + 3e^t$
c) $y'' + 4y' - 13y = 4 + 3\ln t$
d) $y'' + 6y' + 13y = 4 + 3e^t$
e) $y'' - 6y' + 13y = 4 + 3e^t$

3. A linear differential operator that annihilates the function

$$x^2 - x \sin 3x$$

is

a) $D^3(D^2+9)^2$
b) $D^3(D^2+9)$
c) $D^2(D^2+9)^2$
d) $D^3(D^2-9)$
e) $D^3(D^2-9)^2$

4. Two roots of a quadratic auxiliary equation are $m_1 = 1+2i$ and $m_2 = 1-2i$. Then the corresponding 2nd order Homogeneous linear differential equation with constant coefficients is:

a) $y'' - 2y' + 5y = 0$
b) $y'' + 3y = 0$
c) $y'' - 3y' + 4y = 0$
d) $y'' - 2y' + 4y = 0$
e) $y'' - 2y' - 2y = 0$

5. Let $y = c_1 \sin(x^2) + c_2 \cos(x^2)$ be the general solution of the differential equation

$$xy'' - y' + 4x^3 y = 0.$$

The boundary value problem

$$\begin{cases} xy'' - y' + 4x^3 y = 0 \\ y\left(\frac{\sqrt{\pi}}{2}\right) = \sqrt{2},\ y'\left(-\frac{\sqrt{\pi}}{2}\right) = 0. \end{cases}$$

a) has a unique solution satisfying $y(0) = 1$

b) has no solution

c) has infinitly many solutions

d) has a unique solution satisfying $y(0) = 0$

e) has a unique solution satisfying $y(0) = 2$

6. If L is a linear differential-operator such that the particular solution of $Ly = 5x^2 + 3x - 16$ is $y_p = x^2 + 3x$ and for $Ly = -9e^{3x}$ is $y_p = 3e^{2x}$, then a particular solution of $Ly = 18e^{3x} + 10x^2 + 6x - 32$ is

a) $-6e^{2x} + 2x^2 + 6x$

b) $6e^{2x} + 2x^2 + 6x$

c) $-6e^{2x} - 2x^2 - 6x$

d) $-18e^{3x} + 2x^2 + 6x$

e) $18e^{3x} + 2x^2 - 6x$

7. If $\{ae^{2x}, be^{mx}\}$ forms a fundamental set of solutions for a second order linear homogeneous differential equation then (a, b and m are constants)

a) $a \neq 0,\ m \neq 2$

b) $a = 0,\ b = 1$

c) $a \neq 0,\ m = 2$

d) $a = 0,\ b = 3$

e) $a = 0,\ m = 2$

8. **(8 points)** Given that $y_1(x) = x\sin(\ln x)$ is a solution of the differential equation $x^2y'' - xy' + 2y = 0$. Use reduction of order formula to find a second solution $y_2(x)$.

$$y_2(x) = y_1(x)\int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx \qquad (2\text{ pts})$$

The standard form of the DE is

$$y'' - \frac{1}{x}y' + \frac{2}{x^2}y = 0. \qquad (1\text{ pt})$$

$$\therefore\ P(x) = -\frac{1}{x}.$$

$$y_2(x) = x\sin(\ln x)\int \frac{e^{\int \frac{1}{x}dx}}{x^2\sin^2(\ln x)}\,dx \qquad (1\text{ pt})$$

$$= x\sin(\ln x)\int \frac{e^{\ln x}}{x^2\sin^2(\ln x)}\,dx \qquad (1\text{ pt})$$

$$= x\sin(\ln x)\int \frac{x}{x^2\sin^2(\ln x)}\,dx$$

$$= x\sin(\ln x)\int \frac{\csc^2(\ln x)}{x}\,dx$$

$$= x\sin(\ln x)\cdot\left(-\cot(\ln x)\right) \qquad (2\text{ pts})$$

$$= -x\cos(\ln x). \qquad (1\text{ pt})$$

9. **(10 points)** Find the general solution of the differential equation

$$(D^2 - 2D + 5)^2(D^3 + 2D^2 + D + 2)\,y = 0.$$

The auxiliary equation is

$$(m^2 - 2m + 5)^2 (m^3 + 2m^2 + m + 2) = 0 \quad (1\text{pt})$$

$$\Rightarrow (m^2 - 2m + 5)^2 \left[m^2 (m+2) + (m+2) \right] = 0$$

$$\Rightarrow (m^2 - 2m + 5)^2 (m+2)(m^2+1) = 0 \quad (1\text{pt})$$

$$\Rightarrow m + 2 = 0 \Rightarrow m = -2 \quad (1\text{pt})$$

or

$$m^2 + 1 = 0 \Rightarrow m = \pm i \quad (1\text{pt})$$

or

$$(m^2 - 2m + 5)^2 = 0 \Rightarrow m = \frac{2 \pm \sqrt{4 - 4(1)(5)}}{2}$$

$$= \frac{2 \pm 4i}{2} = 1 \pm 2i \text{ (order 2).}$$

(1pt)

So, the general solution is

(1pt) (1pt) (1pt) (1pt)

$$y = c_1 e^{-2x} + c_2 \cos x + c_3 \sin x + (c_4 + c_5 x) e^{x} \cos 2x$$

(1pt)

$$+ (c_6 + c_7 x) e^{x} \sin(2x).$$

10. **(10 points)** Use the method of undetermined coefficients to solve

$$y'' + 4y' + 4y = 2x + 6.$$

First, we solve the homogeneous equation

$$y'' + 4y' + 4 = 0.$$

The auxiliary equation is $m^2 + 4m + 4 = 0$

$\Rightarrow (m+2)^2 = 0 \Rightarrow m = -2$ (order 2). (1pt)

$\therefore\ y_c = (c_1 + c_2 x)e^{-2x}$. (1pt)

Now, $2x+6$ is annihilated by D^2, so (2pts)

$D^2(D^2 + 4D + 4)y = 0$ which has auxiliary equation

$m^2(m^2 + 4m + 4) = 0. \Rightarrow m = 0, 0, -2, -2.$ (1pt)

$y = c_1 + c_2 x + (c_3 + c_4 x)e^{-2x}$ (1pt)

(1pt) → $\therefore\ y_p = A + Bx \Rightarrow y_p' = B \Rightarrow y_p'' = 0$

Substituting in the DE, we get

$$0 + 4B + 4A + 4Bx = 2x + 6 \Rightarrow \begin{cases} 4B + 4A = 6 \\ 4B = 2 \end{cases}$$

(1pt) → $\therefore\ B = \frac{1}{2}$ and $A = 1$ ← (1pt)

The general solution is $y = y_c + y_p$

$$= (c_1 + c_2 x)e^{-2x} + 1 + \frac{x}{2}$$

(2pts)

11. **(12 points)** Use variation of parameters to find a particular solution of the differential equation

$$y'' + 4y = \sin x.$$

First, we solve $y'' + 4y = 0$.

The auxiliary equation is $m^2 + 4 = 0$ (1 pt)

$\Rightarrow m = \pm 2i$ and $y_c = c_1 \cos 2x + c_2 \sin 2x$ (1 pt)

Let $y_p = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$ (1 pt) where

$y_1(x) = \cos 2x$ and $y_2(x) = \sin 2x$.

$$W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2\cos^2(2x) + 2\sin^2(2x) = 2. \quad (1\text{ pt})$$

$$W_1 = \begin{vmatrix} 0 & \sin 2x \\ \sin x & 2\cos 2x \end{vmatrix} = -\sin x \sin 2x \quad (1\text{ pt})$$

$$W_2 = \begin{vmatrix} \cos 2x & 0 \\ -2\sin 2x & \sin x \end{vmatrix} = \sin x \cos(2x) \quad (1\text{ pt})$$

$$u_1 = \int \frac{W_1}{W}\, dx \ (1\text{ pt}) = \int -\frac{1}{2}\sin x \sin 2x\, dx = -\int \sin^2 x \cos x\, dx = -\frac{\sin^3 x}{3} \quad (2\text{ pts})$$

$$u_2 = \int \frac{W_2}{W}\, dx \ (1\text{ pt}) = \int \frac{1}{2}\sin x \cos(2x)\, dx = \int \frac{1}{2}\sin x\,(2\cos^2 x - 1)\, dx$$

$$= \int \left(\sin x \cos^2 x - \frac{1}{2}\sin x\right) dx = -\frac{\cos^3 x}{3} + \frac{1}{2}\cos x \quad (2\text{ pts})$$

$$y_p = -\frac{\sin^3 x}{3}\cos 2x + \left(-\frac{\cos^3 x}{3} + \frac{1}{2}\cos x\right)\sin 2x \quad (1\text{ pt})$$