1. If $y = e^{ax} + e^{bx}$ with $a \neq b$, is a solution of $y'' - 10y' + 16y = 0$. Then $a + b =$

(a) 10

(b) 4

(c) 16

(d) $-8$

(e) $-5$

$$y' = ae^{ax} + be^{bx}$$

$$y'' = a^2 e^{ax} + b^2 e^{bx}$$

$$y'' - 10y' + 16y =$$

$$(a^2 - 10a + 16)e^{ax} + (b^2 - 10b + 16)e^{bx} = 0$$

$$a^2 - 10a + 16 = 0 \quad \& \quad b^2 - 10b + 16 = 0$$

$$(a-8)(a-2) = 0 \qquad (b-2)(b-8) = 0$$

$$a = 8, 2 \qquad b = 2, 8$$

$$\boxed{\Rightarrow a + b = 10}$$

2. If $y$ is the solution of

$$xy' - y = x\ln x, \quad y(1) = 2, \quad (x > 0),$$

then $y(2) =$

(a) $(\ln 2)^2 + 4$
(b) $(\ln 2)^2 - 4$
(c) $(\ln 2)^2 + 3$
(d) $(\ln 2)^2 - 3$
(e) $(\ln 2)^2 + e$

$$y' - \frac{1}{x} y = \ln x$$

I.F. $= e^{-\int \frac{1}{x} dx} = e^{-\ln |x|} = \frac{1}{x}$

$$\frac{d}{dx}\left(\frac{1}{x} y\right) = \frac{\ln x}{x}$$

integrate and let $u = \ln x$

$du = \frac{1}{x} dx$

$$\frac{y}{x} = \int u \, du + C$$

$$\frac{y}{x} = \frac{(\ln x)^2}{2} + C$$

$$y = \frac{x(\ln x)^2}{2} + Cx$$

$$2 = C \Rightarrow y = \frac{x(\ln x)^2}{2} + 2x$$

$$y(2) = (\ln 2)^2 + 4$$

3. The differential equation

M N

$$(xe^x + ay^2 + by)dx - (4xy - x + ye^y)dy = 0$$

is exact, if

$M_y = N_x$

$2ay + b = -4x + 1$

$2a = -4 \rightarrow a = -2$

$b = 1$

(a) $a = -2$ and $b = 1$
(b) $a = -2$ and $b = -1$
(c) $a = 2$ and $b = 1$
(d) $a = 2$ and $b = -1$
(e) $a = -4$ and $b = -1$

4. If $y$ is a solution of

$$y' - y = e^x y^2, \quad y(0) = -1,$$

then $y(1) =$

$$-y^{-2} y' + y^{-1} = -e^x$$

(a) $\dfrac{-2e}{e^2+1}$

(b) $\dfrac{e}{e^2+1}$

(c) $1$

(d) $\dfrac{2e}{e^2+1}$

(e) $\dfrac{e^2}{e-1}$

$$v = y^{-1}$$

$$v' = -y^{-2} y'$$

$$v' + v = -e^x$$

$$\underline{\underline{\text{IF}}} = e^{\int 1\,dx} = e^x$$

$$[e^x v]' = -e^{2x}$$

$$e^x v = -\frac{e^{2x}}{2} + C$$

$$[y]^{-1} = v = -\frac{e^x}{2} + Ce^{-x}$$

$$-1 = -\tfrac{1}{2} + C \Rightarrow C = -1/2$$

$$y^{-1} = \frac{-e^x - e^{-x}}{2} \Rightarrow y = \frac{-2}{e^x + e^{-x}}$$

$$y(1) = \frac{-2}{e + e^{-1}} = \frac{-2e}{e^2+1}$$

5. A small metal bar is dropped into a large container of boiling water (that is the temperature of the medium around the metal bar $T_m = 100^o$C). After 1 second, the temperature of the bar increases to $60^o$C. At $t = 4$ second, the temperature of the bar is measured to be $95^o$C. The initial temperature of the bar equals

(a) $20^o$C
(b) $10^o$C
(c) $30^o$C
(d) $50^o$C
(e) $40^o$C

$$T = 100 + Ce^{kt}$$

$$60 = 100 + Ce^{k}$$

$$95 = 100 + Ce^{4k}$$

$$Ce^{k} = -40$$

$$Ce^{4k} = -5$$

$$e^{3k} = 1/8 \Rightarrow k = \ln 1/2$$

$$T(0) = 20$$

6. (6 points)Find the largest interval containing a unique solution to the following initial values problem ?

$$\sqrt{4-x}\frac{dy}{dx} - \sqrt{1-y} = 0, \quad y(2) = 1/2$$

$$\frac{dy}{dx} = \frac{\sqrt{1-y}}{\sqrt{4-x}} \qquad y(2) = 1/2$$

$f = \dfrac{\sqrt{1-y}}{\sqrt{4-x}}$ is cont if $x < 4$ & $y \leq 1$ 1

$\dfrac{\partial f}{\partial y} = \dfrac{-1}{2\sqrt{(1-y)(4-x)}}$ cont if $x < 4$ & $y < 1$ 1

By existence thm unique soln exists if $y < 1$ & $x < 4$ 2

soln for DE $\sqrt{1-y} = \sqrt{4-x} + C$

$y(2) = 1/2 \Rightarrow C = 1/\sqrt{2} - \sqrt{2} < 0$

so $y < 1$, $x < 4$ & $\sqrt{4-x} - (\sqrt{2} - 1/\sqrt{2}) > 0$ 2

$\Rightarrow x < 4 - 1/2 = 7/2$

$\Rightarrow x < 7/2$ & $y < 1$

7. (10 points) Consider the differential equation

$$y^2 \frac{dy}{dx} = xy^3 - x,$$

(a) Solve the differential equation explicitly by using separation of variables

(b) Find all constant non-singular solution(s)

(a) $\frac{y^2}{(y^3-1)} dy = x\,dx$ 2 $\qquad u = y^3 - 1$, $du = 3y^2\,dy$

$\frac{1}{3} \ln |y^3 - 1| = \frac{x^2}{2} + C$ 2

$\ln |y^3 - 1| = \frac{3x^2}{2} + C$

$y^3 - 1 = k e^{3x^2/2}$ $\qquad k = \pm e^C$

$y^3 = 1 + k e^{3x^2/2}$

$y = \sqrt[3]{1 + k e^{3x^2/2}}$ GS 3

(b) $y = C \Rightarrow 0 = x(C^3 - 1)$ $\forall x$

$\Rightarrow C = 1 \Rightarrow \boxed{y = 1}$ 1

select $k = 0$ in the GS

$\Rightarrow y = 1$

$\Rightarrow y = 1$ 1 nonsing constant solution 1

8. (8 points) Consider the following differential equation

$$\overset{M}{xy^2}\,dx + (\overset{N}{xy} - y)dy = 0, \quad x > 1$$

(a) Show that the above differential equation is not exact

(b) Change the differential equation into exact. (Do not solve it)

ⓐ $M_y = 2xy$ & $N_x = y \Rightarrow$ 2

$M_y \neq N_x$ not exact DE 1

$$\frac{M_y - N_x}{N} = \frac{2xy - y}{y(x-1)} = \frac{2x-1}{x-1} \text{ fn. of } x \quad 1$$

$$\text{IF} \quad e^{\int \frac{2x-1}{x-1}dx} = e^{\int 2 + \frac{1}{x-1}dx} \quad 1$$

$$= e^{2x + \ln(x-1)} = (x-1)e^{2x} \quad 2$$

$$(x-1)e^{2x}\,xy^2\,dx + (xy-y)(x-1)e^{2x}\,dy = 0 \quad 1$$

$$M_y = 2xy(x-1)e^{2x}$$

$$N_x = 2xy(x-1)e^{2x}$$

$\Rightarrow$ exact

9. (10 points) Solve the exact differential equation

$$y^2x(x-1)e^{2x}dx + y(x-1)^2e^{2x}dy = 0$$

$f_x = y^2(x^2-x)e^{2x}$

$f_y = y(x-1)^2e^{2x}$

$f = \dfrac{y^2(x-1)^2e^{2x}}{2} + h(x)$ 2

$f_x = y^2(x-1)e^{2x} + y^2(x-1)^2e^{2x} + h'(x)$

$= y^2x(x-1)e^{2x}$ 4

$y^2(x-1)e^{2x}[\ x\ ] + h'(x)$

$= y^2x(x-1)e^{2x}$

$\Rightarrow h'(x) = 0 \Rightarrow h(x) = C$ 2

Sol $y^2(x-1)^2e^{2x} = C$ 2

10. (6 points) Change the following differential equation into separable:

$$y' = \frac{y\sin(y/x) + x}{x\sin(y/x)}$$

$y' = \frac{y}{x} + \frac{1}{\sin y/x}$ (*) (1)

$v = \frac{y}{x} \Rightarrow xv = y$ (1)

$v + xv' = y'$ (1)

sub in (*)

$v + xv' = v + \frac{1}{\sin v}$ (1)

$x v' = \frac{1}{\sin v}$

$\sin v \, dv = \frac{dx}{x}$ (2)

11. (5 points) Change the following differential equation into separable:

$y' = 1 + e^{y-x+5}$ (a)

$u = y - x + 5$ (1)

$u' = y' - 1$ (1)

sub in (a)

$u' + 1 = 1 + e^u$ (1)

$u' = e^u$

$\frac{du}{e^u} = dx$ (2)

$e^{-u} \, du = dx$