1. A linear differential operator that annihilates the function

$$10 + 7e^{3x}\cos(5x)$$

is

(a) $D(D^2 - 6D + 34)$
(b) $D(D^2 + 6D + 34)$
(c) $D(D^2 - 6D - 25)$
(d) $D(D^2 - 6D - 25)$
(e) $D(D^2 - 6D + 9)$

2. A fourth order linear homogeneous differential equation with constant coefficients has $m_1 = 1 - 7i, \;\; m_2 = m_3 = 0$ as roots of its auxiliary equation. Then this differential equation is

   (a) $y^{(4)} - 2y^{(3)} + 50y'' = 0$

   (b) $y^{(4)} - 2y^{(3)} - 50y'' = 0$

   (c) $y^{(4)} - 2y^{(3)} + 49y'' = 0$

   (d) $y^{(4)} - 2y^{(3)} - 49y'' = 0$

   (e) $y^{(4)} + 2y^{(3)} + 36y'' = 0$

3. Let L be a linear differential operator such that $y_{p1} = xe^{-x}$ and $y_{p2} = x^2 - 8x + 23$ are solutions of the differential equations

$$L(y) = (-x+1)e^{-x}, \quad \text{and} \quad L(y) = x^2 - 2x + 1, \quad \text{respectively.}$$

Find a particular solution of the differential equation

$$L(y) = (2x-2)e^{-x} + (\sqrt{3}x - \sqrt{3})^2$$

(a) $-2xe^{-x} + 3x^2 - 24x + 69$

(b) $2xe^{-x} - 3x^2 + 24x - 69$

(c) $-2xe^{-x} + \sqrt{3}x^2 - 8\sqrt{3}x + 23\sqrt{3}$

(d) $2xe^{-x} + \sqrt{3}x^2 - 8\sqrt{3}x + 23\sqrt{3}$

(e) $xe^{-x} + x^2 - 8x + 23$

4. If $\{a, bx, x^n\}$ forms a fundamental set of solutions for a third order linear homogeneous differential equation (a, b and n are constants). Then

(a) $a \neq 0,\ n \neq 1$

(b) $a = 0,\ b \neq 0$

(c) $a \neq 0,\ b = 0$

(d) $a \neq 0,\ n = 0$

(e) $b \neq 0,\ n = 1$

5. Given $y = c_1 \cos(\ln x) + c_2 \sin(\ln x)$ be the general solution of the differential equation

$$x^2y'' + xy' + y = 0.$$

The initial value problem

$$x^2y'' + xy' + y = 0, \qquad y(e^{\pi/4}) = 2\sqrt{2}, \quad y'(e^{\pi/4}) = 0$$

(a) has a unique solution satisfying $y(e^{\pi/3}) = \sqrt{3} + 1$

(b) has no solution

(c) has infinitly many solutions

(d) has a unique solution satisfying $y(e^{\pi/3}) = \sqrt{3}$

(e) has a unique solution satisfying $y(e^{\pi/3}) = 1$

7. Given that $y_1 = \sin(x^2)$ is a solution of the differential equation

$$xy'' - y' + 4x^3 y = 0, \quad x > 0.$$

Find the general solution.

The equation in standard form is

$$y'' - \frac{1}{x} y' + 4x^2 y = 0. \qquad P(x) = -\frac{1}{x}. \quad 2\text{pts}$$

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{(y_1(x))^2} dx \qquad 2\text{pts}$$

$$= \sin(x^2) \int \frac{e^{\int \frac{1}{x} dx}}{\sin^2(x^2)} dx \qquad 1\text{pt}$$

$$= \sin(x^2) \int \frac{x}{\sin^2(x^2)} dx$$

$$= \sin(x^2) \int x \csc^2(x^2) dx$$

$$= \sin(x^2) \left[-\frac{1}{2} \cot(x^2) \right] \qquad 2\text{pts}$$

$$= -\frac{1}{2} \cos(x^2) \qquad 1\text{pt}$$

The general solution is

$$y = C_1 \sin(x^2) + C_2 \cos(x^2) \qquad 2\text{pts}$$

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8. Find the general solution of the differential equation

$$D^2(D^2 + D + 1)(D^2 + 6D + 9)y = 0.$$

The auxiliary equation is

$$m^2(m^2+m+1)(m^2+6m+9) = 0 \quad \text{2 pts}$$

$$\Rightarrow m = 0, 0, \frac{-1 \pm \sqrt{1-4}}{2}, -3, -3$$

$$\Rightarrow m = \underbrace{0, 0}_{\text{1 pt}}, \underbrace{-\frac{1}{2} \pm \frac{\sqrt{3}}{2} i}_{\text{1 pt}}, \underbrace{-3, -3}_{\text{1 pt}}$$

The general solution is

$$y = \underbrace{c_1 + c_2 x}_{\text{1 pt}} + \underbrace{c_3 e^{-3x} + c_4 x e^{-3x}}_{\text{pts}} + \underbrace{c_5 e^{-\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right) + c_6 e^{-\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right)}_{\text{2 pts}}$$

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9. Given that $y_1 = x$ and $y_2 = x^2$ are solutions of

$$x^2y'' - 2xy' + 2y = 0.$$

Find a particular solution of

$$x^2y'' - 2xy' + 2y = x^2 \ln x.$$

Let $y_p = u_1(x)\,(x) + u_2(x)\,(x^2)$ 2 pts

$$W = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2$$ 1 pt

$$W_1 = \begin{vmatrix} 0 & x^2 \\ \ln x & 2x \end{vmatrix} = -x^2 \ln x$$ 1 pt

$$W_2 = \begin{vmatrix} x & 0 \\ 1 & \ln x \end{vmatrix} = x \ln x$$ 1 pt

$$u_1(x) = \int \frac{W_1}{W}\,dx = \int -\ln x\,dx = x - x\ln x$$ 2 pts

$$u_2(x) = \int \frac{W_2}{W}\,dx = \int \frac{\ln x}{x}\,dx = \frac{(\ln x)^2}{2}$$ 2 pts

$$\therefore\ y_p = x^2 - x^2 \ln x + \frac{x^2}{2}(\ln x)^2$$ 1 pt

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10. Use the method of undetermined coefficients to solve

$$y'' + 4y' + 3y = 24e^x - 6x$$

$y'' + 4y' + 3y = 0$. The auxiliary equation is

$m^2 + 4m + 3 = 0 \Rightarrow (m+3)(m+1) = 0 \Rightarrow m = -3, -1$

$y_c = c_1 e^{-3x} + c_2 e^{-x}$ 1pt

An annihilator of $24e^x - 6x$ is $(D-1)D^2$ 2pts

Now, apply $(D-1)(D^2)$ to both sides to get

$(D-1)(D^2)(D+3)(D+1)y = 0$.

The solution is

$$y = \underbrace{c_1 e^{-3x} + c_2 e^{-x}}_{y_c} + \underbrace{c_3 + c_4 x + c_5 e^x}_{y_p}$$ 2pts

$\therefore y_p = A + Bx + Ce^x$ 1pt

$y_p' = B + Ce^x \Rightarrow y_p'' = Ce^x$

Substituting in the DE gives

$Ce^x + 4B + 4Ce^x + 3A + 3Bx + 3Ce^x = 24e^x - 6x$

$\therefore 8C = 24 \Rightarrow C = 3$ 1pt

$\Rightarrow A = \frac{8}{3}$ 1pt

$3B = -6 \Rightarrow B = -2$ 1pt

The general solution is $y = c_1 e^{-3x} + c_2 e^{-x} + 3e^x - 2x + \frac{8}{3}$ 1pt

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