King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 202
Final Exam, Semester II, 2021-2022
Net Time Allowed: 150 minutes

Name:____

ID:____________________Section:__________Serial:________

1. Make sure you have 12 MCQ and 3 written questions.
2. The weight for each MCQ is 6 marks.
3. For the MCQ use the OMR sheets for the answers only.
4. The table below is for written questions only.

Q#	Marks	Maximum Marks
13		13
14		10
15		10
Total		33

1. Write clearly.
2. Show all your steps.
3. No credit will be given to wrong steps.
4. Do not do messy work.
5. Calculators and mobile phones are NOT allowed in this exam.
6. Turn off your mobile.

1. Use the substitution $x = e^t$ to transform the Cauchy-Euler equation

$$x^2y'' + 10xy' + 8y = 0,$$

to a differential equation with constant coefficients of the form

$$A\frac{d^2y}{dt^2} + B\frac{dy}{dt} + Cy = 0.$$

Then $A + B + C =$

(a) 18

(b) 19

(c) 20

(d) 17

(e) 16

2. The function $1 + 2x - e^{2x} - 5x\sin(2x)$ is annihilated by

(a) $D^2(D-2)(D^2+4)^2$

(b) $D^2(D-2)(D^2-4)^2$

(c) $D^2(D-2)^2(D^2+4)$

(d) $D(D-2)(D^2+4)^5$

(e) $D^2(D-2)^2(D^2-4)^2$

3. Given that $X_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{2t}$ are two linearly independent solutions of $X' = AX$. If

$$X = X_1 + \frac{2}{3} X_2 + X_p,$$

is the solution of

$$X' = AX + \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad X(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

then $X_p(1) =$

(a) $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 5 \\ 7 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(d) $\begin{bmatrix} 6 \\ 8 \end{bmatrix}$

(e) $\begin{bmatrix} 11 \\ -3 \end{bmatrix}$

4. Consider the following system of initial value problem

$$X' = AX, \quad X(0) = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

where $A = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix}$. Then $X(1) =$

Hint: $\lambda_1 = \lambda_2 = 5$ are repeated eigenvalues of A.

(a) $\begin{bmatrix} e^5 \\ -3e^5 \end{bmatrix}$

(b) $\begin{bmatrix} e^5 \\ e^5 \end{bmatrix}$

(c) $\begin{bmatrix} -2e^5 \\ 3e^5 \end{bmatrix}$

(d) $\begin{bmatrix} 7e^5 \\ 5e^5 \end{bmatrix}$

(e) $\begin{bmatrix} -5e^5 \\ -4e^5 \end{bmatrix}$

5. The solution $y(x)$ of the following exact differential equation

$$(x+y)^2 dx + (2xy + x^2 - 1)dy = 0, \quad y(1) = 1$$

at $x = 0$ is equal to

(a) $-4/3$

(b) $4/3$

(c) -1

(d) 1

(e) 0

6. Given that $y_1 = x \sin x$ is a solution of

$$x^2 y'' - 2xy' + (x^2 + 2)y = 0, \quad x > 0,$$

then a second solution y_2 is

(a) $-x \cos x$
(b) $-(x^2 + 2) \cos x$
(c) $-x^2 \sin x$
(d) $-(x^2 + 2) \sin x$
(e) $-x^2 \cos x$

7. The indicial roots about the singular point $x_0 = 0$ of the differential equation

$$4x^2y'' - 4x^2y' + (1 - 2x)y = 0, \text{ are}$$

(a) $r = 1/2$ repeated

(b) $r = -1/2$ repeated

(c) $r = 1/2$ and $r = -1/2$

(d) $r = 3/2$ repeated

(e) $r = 3/2$ and $r = -3/2$

8. The solution $y(x)$ of the second order initial value problem

$$y'' - 2y' + 5y = 0, \quad y(0) = 2,\ y'(0) = 6,$$

at $x = \dfrac{\pi}{8}$ is

(a) $2\sqrt{2}\, e^{\pi/8}$

(b) $3\sqrt{2}\, e^{\pi/8}$

(c) $4\sqrt{2}\, e^{\pi/8}$

(d) $5\sqrt{2}\, e^{\pi/8}$

(e) $6\sqrt{2}\, e^{\pi/8}$

9. By using the substitution $u = y/x$, the differential equation

$$xy + y^2 + x^2 - x^2\frac{dy}{dx} = 0, \quad x > 0,$$

can be transformed into

(a) $\dfrac{du}{1+u^2} = \dfrac{dx}{x}$

(b) $\dfrac{du}{1-u^2} = \dfrac{dx}{x}$

(c) $\dfrac{du}{1+2u^2} = \dfrac{dx}{x}$

(d) $\dfrac{du}{u^2-1} = \dfrac{dx}{x^2}$

(e) $\dfrac{du}{1+u^2} = \dfrac{dx}{x^2+1}$

10. If $y = \sum_{n=0}^{\infty} c_n x^n$ is a power series solution about the ordinary point $x_0 = 0$ of the differential equation $y'' - xy' + 4y = 0$, then the coefficients c_n satisfy

(a) $c_{n+2} = \dfrac{(n-4)}{(n+2)(n+1)} c_n, \quad n \geq 1$

(b) $c_{n+2} = \dfrac{(n+2)}{(n+1)(n+3)} c_n, \quad n \geq 1$

(c) $c_{n+2} = \dfrac{(n+1)}{n(n+2)} c_n, \quad n \geq 1$

(d) $c_{n+2} = \dfrac{(4-n)}{(n+2)(n+3)} c_n, \quad n \geq 1$

(e) $c_{n+2} = \dfrac{(n+2)}{n(n+3)} c_n, \quad n \geq 1$

11. Without actually solving the differential equation, find the minimum radius of convergence of a power series solution for

$$(1 + x + x^2)y'' - 3y = 0, \quad \text{about } x_0 = 1 \text{ is}$$

(a) $\sqrt{3}$

(b) $\sqrt{5}$

(c) $\sqrt{2}$

(d) $\sqrt{7}$

(e) $\sqrt{11}$

12. The sum of all regular singular points of the differential equation

$$(x+2)^2(x^2-1)y'' + 2xy' + 6y = 0,$$

is

(a) 0

(b) 1

(c) -1

(d) 2

(e) $-1/2$

13. Solve $X' = AX$ where

$$A = \begin{bmatrix} -2 & 1 & 3 \\ 0 & 1 & -4 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow \det(A-\lambda I) = \begin{vmatrix} -2-\lambda & 1 & 3 \\ 0 & 1-\lambda & -4 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0 \quad (1)$$

$$= (-2-\lambda)\left[(1-\lambda)^2+4\right] = 0$$

$$= -(2+\lambda)(\lambda^2-2\lambda+5) = 0$$

$$\lambda = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i, \quad \lambda = -2 \quad (2)$$

for $\lambda = -2$:

$$\begin{bmatrix} 0 & 1 & 3 \\ 0 & 3 & -4 \\ 0 & 1 & 3 \end{bmatrix} \rightarrow (1)\ v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow (1)\ X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{-2t}$$

$\lambda = 1-2i$:

$$\begin{bmatrix} -3+2i & 1 & 3 \\ 0 & 2i & -4 \\ 0 & 1 & 2i \end{bmatrix} \rightarrow \begin{bmatrix} -3+2i & 0 & 3-2i \\ 0 & 0 & 0 \\ 0 & 1 & 2i \end{bmatrix}$$

$$v = \begin{bmatrix} 1 \\ -2i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} = B_1 + i B_2 \quad (2)$$

$$X_2 = e^{t}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cos 2t - \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} \sin 2t \right\} = (2) \begin{bmatrix} \cos 2t \\ 2\sin 2t \\ \cos 2t \end{bmatrix} e^{t}$$

$$X_3 = e^{t}\left\{ \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} \cos 2t + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \sin 2t \right\} = \begin{bmatrix} \sin 2t \\ -2\cos 2t \\ \sin 2t \end{bmatrix} \quad (2)$$

GS: $X = c_1 X_1 + c_2 X_2 + c_3 X_3 \quad (2)$

14. Let

$$A = \begin{bmatrix} -1 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

(a) Show that $A^3 = \mathbf{0}$.

(b) Find e^{At}

(c) Solve $X' = AX$

$$A^2 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad (2)$$

$$A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (1)$$

(b) $$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots + \frac{A^nt^n}{n!} + \cdots \quad (2)$$

$$e^{At} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} t + \frac{t^2}{2} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad (1)$$

$$e^{At} = \begin{bmatrix} 1 - t - \frac{t^2}{2} & t & t + \frac{t^2}{2} \\ -t & 1 & t \\ -t - \frac{t^2}{2} & t & 1 + t + \frac{t^2}{2} \end{bmatrix} \quad (2)$$

(c) $$X = e^{At}C = \begin{bmatrix} 1 - t - \frac{t^2}{2} & t & t + \frac{t^2}{2} \\ -t & 1 & t \\ -t - \frac{t^2}{2} & t & 1 + t + \frac{t^2}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

(1) (1)

15. Find the first three nonzero terms of the series solution $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ of the equation $4xy'' + \frac{1}{2}y' + y = 0$ corresponds to the largest indicial root $r = \frac{7}{8}$.

$$y = \sum_{n=0}^{\infty} c_n x^{n+r} \Rightarrow y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} \quad (2)$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2} \quad (1)$$

$$4\sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-1} + \frac{1}{2}\sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} + \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\left(4r^2 - \frac{7}{2}r\right) c_0 x^{r-1} + \sum_{k=1}^{\infty} \Big[4(k+r)(k+r-1)c_k + \frac{1}{2}(k+r)c_k + c_{k-1}\Big] x^{k+r} = 0$$

$$\frac{1}{2}(k+r)(8k+8r-7)c_k + c_{k-1} = 0 \quad (2)$$

$$r = 7/8 \Rightarrow c_k = -\frac{2c_{k-1}}{(8k+7)} \quad (2)$$

$k = 1, 2, 3, \ldots$

$$c_1 = -\frac{2}{15}c_0 \; (1); \quad c_2 = \frac{2}{345}c_0 \; (1), \quad c_3 = \frac{-4}{32085}c_0$$

$$y_1 = x^{7/8}\left(1 - \frac{2}{15}x + \frac{2}{345}x^2 - \cdots\right) \quad (1)$$