King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 208 - Major Exam I - Term 211

Duration: 90 minutes

1. The solution of the initial value problem:

$$\frac{dy}{dx} = \frac{y}{x+y^2} \quad y(2) = 1$$

is equal to

1. $y^2 + y - x = 0$
2. $y^2 + y + x = 4$
3. $y^2 - y - x = -2$
4. $y^2 + y + 2x = 6$
5. $y^2 - y - 2x = -4$

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2. Consider the differential equation

$$xy' + \frac{y-1}{3} = \frac{1}{3(y-1)^2}$$

Given that its general solution is:

$$\ln|1-(y-1)^3| = \ln|\frac{1}{x}| + C$$

Then: ***(choose the true statement)***

1. the singular solution is $y = 2$ only.
2. the singular solution is $y = -1$ only.
3. the singular solutions are $y = 1$ and $y = -1$.
4. there is no singular solutions.
5. the singular solution is $y = \frac{1}{3}$.

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3. In a certain culture of bacteria, the number of bacteria was initially 1000. After two hours, the population became 1500. Assuming that the population of bacteria grows at a rate proportional to the number of bacteria presented at time t, then **the time t when the bacteria population will be 2250 equals to**

(calculational tip: $(15)^2 = 225$*)*

1. 4 hours.
2. 3.5 hours.
3. 3 hours.
4. 4.25 hours.
5. 2.5 hours.

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4. Let A and B be two constants in the differential equation

$$[Axy + (A+B)y\sin 2x]dx + [x^2 + B\cos^2 x]dy = 0.$$

Then the values of A and B that make the differential equation **exact** are

1. $A = 2$, $B = -1$.
2. $A = 1$, $B = 1$.
3. $A = -1$, $B = -1$.
4. $A = 1$, $B = -2$.
5. $A = 1$, $B = \frac{-1}{2}$.

5. Let c_1 and c_2 be arbitrary constants. Verify that $y = c_1 x + c_2 x \ln x$ is a solution of the differential equation

$$x^2 y'' - xy' + y = 0.$$

$$y = c_1 x + c_2 x \ln x$$

$$y' = c_1 + c_2 \ln x + c_2$$

$$y'' = \frac{c_2}{x}$$

$$\text{LHS} = x^2 y'' - xy' + y = x^2\left(\frac{c_2}{x}\right) - x\left(c_1 + c_2 \ln x + c_2\right) + c_1 x + c_2 x \ln x$$

$$= \underline{c_2 x} - \underline{c_1 x} - c_2 x \ln x - \underline{c_2 x} + \underline{c_1 x} + c_2 x \ln x$$

$$= 0 = \text{RHS}$$

$\Rightarrow y = c_1 x + c_2 x \ln x$ is a solution of the differential equation.

6. Solve the differential equation:

$$xy' - 2y = x^3 e^x, \quad (x > 0)$$

- $y' - \frac{2}{x} y = x^2 e^x$
- $\rho(x) = e^{\int \frac{-2}{x} dx} = e^{-2\ln x} = \frac{1}{x^2}$
- $d\left(y \cdot \frac{1}{x^2}\right) = e^x dx$

$\Rightarrow y \cdot \frac{1}{x^2} = e^x + c$

(OR: $y = x^2 e^x + cx^2$)

7. Find the one-parameter family of solutions of the following differential equation:

$$y \sec x \, \frac{dy}{dx} = xy^2 - x - 1 + y^2$$

$$y \sec x \, \frac{dy}{dx} = x(y^2-1) + (y^2-1)$$

$$\Rightarrow \quad y \sec x \, \frac{dy}{dx} = (x+1)(y^2-1)$$

$$\Rightarrow \quad \frac{y}{y^2-1} \, dy = (x+1) \cos x \, dx$$

$$\Rightarrow \frac{1}{2} \ln |y^2-1| = (x+1)\sin x + \cos x + C$$

$x+1$	$\cos x$
1	$\sin x$
0	$-\cos x$

8. Solve the **exact** differential equation:

$$(2xy + y - \tan y)\, dx + (x^2 - x\tan^2 y + \sec^2 y)\, dy = 0.$$

*(Do **NOT** verify that the differential equation is exact.)*

- $F_x = M \Rightarrow F = \int M\,dx$

$\Rightarrow F = \int (2xy + y - \tan y)\,dx$

$\Rightarrow F = x^2 y + yx - (\tan y)\,x + g(y)$

- $F_y = N \Rightarrow \underline{x^2} + x - x\sec^2 y + g'(y) = \underline{x^2} - x\tan^2 y + \sec^2 y$

$\Rightarrow x\underbrace{(1 - \sec^2 y)}_{= -\tan^2 y} + g'(y) = -x\tan^2 y + \sec^2 y$

$\Rightarrow g'(y) = \sec^2 y$

$\Rightarrow g(y) = \tan y$

$\Rightarrow$ The G.S. is $\boxed{x^2 y + yx - (\tan y)\,x + \tan y = C}$

(could start with $F_y = N \Rightarrow F = \int N\,dy \ldots$)