King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 208 - Major Exam II - Term 211

Duration: 90 minutes

1. If the rank of the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & k^2-3k+4 & 3 \\ 1 & 2 & k^2-5k+8 & 3 \end{bmatrix}$ is equal to 2, then:

1. $k = 2$
2. $k = 1$ or $k = 2$ or $k = 3$
3. $k = 1$ or $k = 2$
4. $k = 2$ or $k = 3$

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2. The function $y = c_1 + e^{2x}(c_2 \cos x + c_3 \sin x)$ is the general solution of:

1. $y^{(3)} - 4y'' + 5y' = 0$
2. $y^{(3)} - 4y'' - 5y = 0$
3. $y^{(2)} - 4y' - 5y = 0$
4. $y^{(3)} - 4y' - 5y = 0$
5. $y^{(3)} + 2y'' - 3y' + 10y = 0$

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3. Consider the following subsets of $\mathbb{R}^3$:

I. $W_1 = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x + y + z = 0\}$

II. $W_2 = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x^2 + y^2 + z^2 = 0\}$

III. $W_3 = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x + y + z \geq 0\}$

IV. $W_4 = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x + y + z = 1\}$

The subsets that are **subspaces** are:

1. I and II.
2. I and III.
3. II and IV.
4. III and IV.
5. II and III

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4. Let a, b, and c be some constants. Consider the linear system:

$$\begin{array}{ccccccc} x & + & y & + & az & = & 1 \\ 5x & - & y & + & bz & = & -1 \\ & & y & + & cz & = & 0 \end{array}$$

If the determinant of the coefficient matrix is 3, then:

1. $z = 2$.
2. $z = \frac{-1}{2}$.
3. $z = -3$.
4. $z = 1$.
5. $z = -6$.

5. Find a basis **and** the dimension of the solution space of the following linear system:

$$\begin{array}{rcrcrcrcl} x & - & 3y & - & 10z & + & 5w & = & 0 \\ x & + & 4y & + & 11z & - & 2w & = & 0 \\ x & + & 3y & + & 8z & - & w & = & 0 \end{array}$$

$$[A\ \ b] = \left[\begin{array}{cccc|c} 1 & -3 & -10 & 5 & 0 \\ 1 & 4 & 11 & -2 & 0 \\ 1 & 3 & 8 & -1 & 0 \end{array}\right]$$

$$\xrightarrow[-R_1+R_3]{-R_1+R_2} \left[\begin{array}{cccc|c} 1 & -3 & -10 & 5 & 0 \\ 0 & 7 & 21 & -7 & 0 \\ 0 & 6 & 18 & -6 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{7}R_2} \left[\begin{array}{cccc|c} 1 & -3 & -10 & 5 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 6 & 18 & -6 & 0 \end{array}\right]$$

$$\xrightarrow{-6R_2+R_3} \left[\begin{array}{cccc|c} 1 & -3 & -10 & 5 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

z, w free, $y = -3z + w$, $x = 3y + 10z - 5w = z - 2w$

a typical vector in the solution space is:

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} z-2w \\ -3z+w \\ z \\ w \end{bmatrix} = z\begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \end{bmatrix} + w\begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \text{Basis} = \left\{ \begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

dimension = 2

6. **[This question has two parts]**
Consider the IVP
$$y^{(3)} - y'' + y' - y = 0$$
$$y(0) = 1, \quad y'(0) = 2, \quad y''(0) = 3$$
The DE above has the solutions: $y_1 = e^x$, $y_2 = \cos x$, and $y_3 = \sin x$.

(a) Prove that the functions y_1, y_2, and y_3 are linearly independent.

(b) Solve the IVP.

(a)
$$W(y_1, y_2, y_3) = \begin{vmatrix} e^x & \cos x & \sin x \\ e^x & -\sin x & \cos x \\ e^x & -\cos x & -\sin x \end{vmatrix}$$

$$= e^x \begin{vmatrix} -\sin x & \cos x \\ -\cos x & -\sin x \end{vmatrix} - e^x \begin{vmatrix} \cos x & \sin x \\ -\cos x & -\sin x \end{vmatrix} + e^x \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$= 2e^x \neq 0 \Rightarrow y_1, y_2, y_3$ are linearly independent.

(b) G.S.: $y = c_1 y_1 + c_2 y_2 + c_3 y_3 = c_1 e^x + c_2 \cos x + c_3 \sin x$

$y' = c_1 e^x - c_2 \sin x + c_3 \cos x$

$y'' = c_1 e^x - c_2 \cos x - c_3 \sin x$

$$\left.\begin{array}{ll} y(0) = 1 \Rightarrow & 1 = c_1 + c_2 \\ y'(0) = 2 \Rightarrow & 2 = c_1 + c_3 \\ y''(0) = 3 \Rightarrow & 3 = c_1 - c_2 \end{array}\right\} \Rightarrow c_1 = 2, \; c_2 = -1, \; c_3 = 0$$

$\Rightarrow$ The solution of the IVP is: $y = 2e^x - \cos x$

7. Solve the differential equation: $y^{(3)} - y' = 1$.

- <u>Finding y_c:</u>

The characteristic eqn: $r^3 - r = 0 \Rightarrow r(r^2-1) = 0$

$$\Rightarrow \begin{cases} r = 0, & m = 1 \\ r = 1, & m = 1 \\ r = -1, & m = 1 \end{cases}$$

$$\Rightarrow y_c = c_1 + c_2 e^{x} + c_3 e^{-x}.$$

- <u>Finding y_p:</u> $F(x) = 1 \Rightarrow a + ib = 0$, a root of $m = 1$

$$\Rightarrow y_p = x \cdot (A)$$

- $y_p = Ax \Rightarrow y_p' = A,\ y_p'' = y_p''' = 0 \overset{\text{in DE}}{\Rightarrow} 0 - A = 1 \Rightarrow A = -1$

$$\Rightarrow y_p = -x$$

- The G.S. $y = y_c + y_p = c_1 + c_2 e^{x} + c_3 e^{-x} - x.$

8. Consider the differential equation:

$$(1-x)y'' + xy' - y = 2(1-x)^2 e^x \quad (0 < x < 1)$$

If the solution of the associated homogeneous differential equation $y_c = c_1 x + c_2 e^x$, then find **the general solution** of the given differential equation.

- The standard form of the DE is : $y'' + \frac{x}{1-x} y' - \frac{1}{1-x} y = 2(1-x)e^x$

- $y_1 = x,\ y_2 = e^x,\ f(x) = 2(1-x)e^x.$

- $W(y_1, y_2) = \begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix} = xe^x - e^x = e^x(x-1)$

- $y_p = -y_1 \int \frac{y_2 f}{W} + y_2 \int \frac{y_1 f}{W}$

- $\int \frac{y_2 f}{W} = \int \frac{e^x \cdot 2(1-x)e^x}{e^x(x-1)} dx = -2\int e^x dx = -2e^x.$

- $\int \frac{y_1 f}{W} = \int \frac{x \cdot 2(1-x)e^x}{e^x(x-1)} dx = -2\int x\,dx = -x^2$

$\Rightarrow y_p = -x(-2e^x) + e^x(-x^2) = e^x(2x - x^2)$

$\Rightarrow$ G.S. $= y_c + y_p$

$= c_1 x + c_2 e^x + e^x(2x - x^2)$