1. The **sum** of all constant solutions of the differential equation

$$\frac{dy}{dx} = y(y^2 - 1)$$

is

(a) 0 (correct)
(b) 3
(c) 2
(d) 4
(e) -1

2. The **sum** of all values of c such that $y = \dfrac{c}{1+x^2}$ is a solution of the differential equation $\dfrac{dy}{dx} = 2xy^2$ is

(a) -1 (correct)
(b) 1
(c) 0
(d) 2
(e) 3

3. An explicit particular solution of the initial value problem

$$\frac{dy}{dx} = 3x^2(y^2+1),\ y(0) = 1$$

is

(a) $y = \tan\left(x^3 + \frac{\pi}{4}\right)$ (correct)

(b) $y = \tan\left(2x^3 + \frac{\pi}{4}\right)$

(c) $y = \tan\left(3x^3 + \frac{\pi}{4}\right)$

(d) $y = \tan\left(4x^3 + \frac{\pi}{4}\right)$

(e) $y = \tan\left(x^3 - \frac{\pi}{4}\right)$

4. In a certain culture of bacteria, the initial population was 5, and it increased to be 40 in 15 hours. How long did it take for the population to be 10?

(a) 5 hours (correct)

(b) 4 hours

(c) 3 hours

(d) 2 hours

(e) 1 hour

5. The matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 4 \\ 4 & k & 2 \end{bmatrix}$ is invertible if $k \neq$

(a) 3 (correct)

(b) 2

(c) 0

(d) -6

(e) 1

6. The values of a and b that make the differential equation

$$(ye^{ay} + bxy)\,dx + ((1+2y)e^{ay}x + 3x^2)\,dy = 0$$

exact are:

(a) $a = 2,\ b = 6$ (correct)

(b) $a = 2,\ b = 3$

(c) $a = 2,\ b = 2$

(d) $a = 6,\ b = 3$

(e) $a = 3,\ b = 6$

7. **(10 points)** Given the system

$$\begin{cases} x + 2y = 3 \\ 2x - y + z = -1 \\ 4x - y + 2z = 2 \end{cases}$$

Use Cramer's Rule to find the value of x.

$$\begin{vmatrix} 1 & 2 & 0 \\ 2 & -1 & 1 \\ 4 & -1 & 2 \end{vmatrix} = 1\begin{vmatrix} -1 & 1 \\ -1 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix}$$

$$= (-2+1) - 2(4-4) \quad (4\text{pts})$$

$$= -1$$

$$\begin{vmatrix} 3 & 2 & 0 \\ -1 & -1 & 1 \\ 2 & -1 & 2 \end{vmatrix} = 3\begin{vmatrix} -1 & 1 \\ -1 & 2 \end{vmatrix} - 2\begin{vmatrix} -1 & 1 \\ 2 & 2 \end{vmatrix}$$

$$= 3(-2+1) - 2(-2-2) \quad (4\text{pts})$$

$$= -3 + 8 = 5$$

$$X = \frac{\begin{vmatrix} 3 & 2 & 0 \\ -1 & -1 & 1 \\ 2 & -1 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 0 \\ 2 & -1 & 1 \\ 4 & -1 & 2 \end{vmatrix}} = \frac{5}{-1} = -5 \quad (2\text{pts})$$

8. **(10 points)** Solve the initial value problem

$$xy' - 3y = x^4 \cos x,\ y(2\pi) = 0.$$

The DE is linear. We put it in standard form

$$y' - \frac{3}{x} y = x^3 \cos x \quad \cdots (1) \qquad (1pt)$$

integrating factor $u(x) = e^{\int -\frac{3}{x} dx} = e^{-3\ln x} = \frac{1}{x^3}, \ x > 0$ (2pts)

We multiply Eq(1) by $\frac{1}{x^3}$ to get

$$\frac{d}{dx}\left(\frac{1}{x^3} y\right) = \cos x \qquad (2pts)$$

$$\Rightarrow \int \frac{d}{dx}\left(\frac{y}{x^3}\right) dx = \int \cos x \, dx$$

(1pt) (1pt) (1pt)

$$\Rightarrow \frac{y}{x^3} = \sin x + C \quad \text{or } y = x^3 \sin x + C x^3$$

$y(2\pi) = 0 \Rightarrow C = 0.$ (1pt)

The solution of the IVP is

$$y = x^3 \sin x \qquad (1pt)$$

9. **(12 points)** Let $A = \begin{bmatrix} 2 & 4 & 2 \\ 4 & 11 & 8 \\ 2 & 5 & 4 \end{bmatrix}$.

Find the inverse of A using the adjoint formula.

$$|A| = \begin{vmatrix} 2 & 4 & 2 \\ 4 & 11 & 8 \\ 2 & 5 & 4 \end{vmatrix} = 2\begin{vmatrix} 11 & 8 \\ 5 & 4 \end{vmatrix} - 4\begin{vmatrix} 4 & 8 \\ 2 & 4 \end{vmatrix} + 2\begin{vmatrix} 4 & 11 \\ 2 & 5 \end{vmatrix}$$

$$= 2(44-40) - 4(16-16) + 2(20-22) = 8-4 = 4$$ (2pts)

$$A_{11} = \begin{vmatrix} 11 & 8 \\ 5 & 4 \end{vmatrix} = 4, \quad A_{12} = -\begin{vmatrix} 4 & 8 \\ 2 & 4 \end{vmatrix} = 0, \quad A_{13} = \begin{vmatrix} 4 & 11 \\ 2 & 5 \end{vmatrix} = -2$$

$$A_{21} = -\begin{vmatrix} 4 & 2 \\ 5 & 4 \end{vmatrix} = -6, \quad A_{22} = \begin{vmatrix} 2 & 2 \\ 2 & 4 \end{vmatrix} = 4, \quad A_{23} = -\begin{vmatrix} 2 & 4 \\ 2 & 5 \end{vmatrix} = -2$$

$$A_{31} = \begin{vmatrix} 4 & 2 \\ 11 & 8 \end{vmatrix} = 10, \quad A_{32} = -\begin{vmatrix} 2 & 2 \\ 4 & 8 \end{vmatrix} = -8, \quad A_{33} = \begin{vmatrix} 2 & 4 \\ 4 & 11 \end{vmatrix} = 6$$

$$[A_{ij}] = \begin{bmatrix} 4 & 0 & -2 \\ -6 & 4 & -2 \\ 10 & -8 & 6 \end{bmatrix} \Rightarrow [A_{ij}]^T = \begin{bmatrix} 4 & -6 & 10 \\ 0 & 4 & -8 \\ -2 & -2 & 6 \end{bmatrix}$$ (8pts)

$$A^{-1} = \frac{[A_{ij}]^T}{|A|} = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{5}{2} \\ 0 & 1 & -2 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$$ (2pts)

10. **(13 points)** Solve the exact differential equation

$$(2xy\cos(x^2) + x)\,dx + \left(\frac{y^2}{1+y} + \sin(x^2)\right)\,dy = 0.$$

The DE is exact $\Rightarrow$ there is a function $f(x,y)$ such that

$$\frac{\partial f}{\partial x} = 2xy\cos(x^2) + x \text{ and } \frac{\partial f}{\partial y} = \frac{y^2}{1+y} + \sin(x^2). \quad (2\text{pts})$$

$$\frac{\partial f}{\partial x} = 2xy\cos(x^2) + x \Rightarrow \int \frac{\partial f}{\partial x}\,dx = \int \left(2xy\cos(x^2) + x\right)dx \quad (1\text{pt})$$

$$\Rightarrow f(x,y) = \underbrace{y\sin(x^2)}_{(1\text{pt})} + \underbrace{\frac{x^2}{2}}_{(1\text{pt})} + \underbrace{g(y)}_{(1\text{pt})}$$

$$\frac{\partial f}{\partial y} = \frac{y^2}{1+y} + \sin(x^2)$$

$$\Rightarrow \cancel{\sin(x^2)} + g'(y) = \frac{y^2}{1+y} + \cancel{\sin(x^2)} \quad (1\text{pt})$$

$$\Rightarrow g(y) = \int \frac{y^2}{y+1}\,dy = \int \left(y - 1 + \frac{1}{y+1}\right)dy \quad (1\text{pt})$$

$$= \frac{y^2}{2} - y + \ln|y+1| \quad (3\text{pts})$$

$$\therefore f(x,y) = y\sin(x^2) + \frac{x^2}{2} + \frac{y^2}{2} - y + \ln|y+1|$$

The solution of the DE is

$$y\sin(x^2) + \frac{x^2}{2} + \frac{y^2}{2} - y + \ln|y+1| = C \quad (2\text{pts})$$