1. If the vector $v = (4, -2)$ is written as $c_1(1, 1) + c_2(-1, 1)$, then $2c_1 - c_2 =$

(a) 5 (correct)

(b) 4

(c) 0

(d) 3

(e) 1

2. The rank of the matrix $A = \begin{bmatrix} 1 & 3 & 3 & 9 \\ 2 & 7 & 4 & 8 \\ 2 & 7 & 5 & 12 \\ 2 & 8 & 3 & 2 \end{bmatrix}$ is

(a) 3 (correct)

(b) 2

(c) 1

(d) 4

(e) 0

3. An appropriate form of a particular solution y_p for the differential equation

$$(D-1)(D+2)(D-3)y = x - 2xe^{-3x}$$

is

(a) $A + Bx + Ce^{-3x} + Exe^{-3x}$ (correct)

(b) $Bx + Exe^{-3x}$

(c) $Ax + Bx^2 + Ce^{-3x} + Exe^{-3x}$

(d) $A + Bx + Cxe^{-3x} + Ex^2e^{-3x}$

(e) $Ax + Bx^2 + Cxe^{-3x} + Ex^2e^{-3x}$

4. Let $W = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1x_3 = 0\}$.
Which one of the following statements is True?

(a) W is not closed under addition but closed under scalar multiplication (correct)

(b) W is a subspace of $\mathbb{R}^3$

(c) W is closed under addition but not closed under scalar multiplication

(d) W is not closed under addition and not closed under scalar multiplication

(e) $W = \phi$

5. The graph of the solution of the Initial Value Problem

$$y'' - y = 0, \; y(0) = 0, \; y'(0) = 2$$

passes through the point

(a) $\left(\ln 2, \frac{3}{2}\right)$ (correct)

(b) $\left(\ln 2, \frac{1}{2}\right)$

(c) $\left(\ln 3, \frac{3}{2}\right)$

(d) $\left(\ln 3, \frac{1}{2}\right)$

(e) $\left(\ln 2, -\frac{1}{2}\right)$

6. A linear homogeneous differential equation with the following solution

$$y = Ae^{2x} + B\cos(2x) + C\sin(2x)$$

is

(a) $y''' - 2y'' + 4y' - 8y = 0$ (correct)

(b) $y''' + 2y'' - 4y' + 8y = 0$

(c) $y''' + 2y'' - 4y' - 8y = 0$

(d) $y''' - 2y'' + 4y' + 8y = 0$

(e) $y''' - 2y'' - 4y' + 8y = 0$

7. **(10 points)** Determine whether or not the vectors

$$v_1 = (1,1,2),\ v_2 = (2,2,0),\ v_3 = (3,4,-1)$$

form a basis for $\mathbb{R}^3$. Justify your answer

We check wether the three vectors are linearly independent.

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 2 & 0 & -1 \end{vmatrix} = 1\begin{vmatrix} 2 & 4 \\ 0 & -1 \end{vmatrix} - 2\begin{vmatrix} 1 & 4 \\ 2 & -1 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} \quad (2 \text{ pts})$$

$$= (-2) - 2(-9) + 3(-4)$$

$$= -2 + 18 - 12$$

$$= 4 \neq 0 \quad (2 \text{ pts})$$

(2 pts)

$\therefore$ V_1, V_2 and V_3 are linearly independent and $\dim \mathbb{R}^3 = 3$, so they form a basis for $\mathbb{R}^3$. (2 pts) (2 pts)

8. **(10 points)** Find the general solution of $(D-1)(D+2)^2(D^2+9)^2y = 0$.

The auxiliary equation is

$$(m-1)(m+2)^2(m^2+9)^2 = 0 \quad (2\text{pts})$$

$$\Rightarrow m = 1, -2, -2, \pm 3i, \pm 3i \quad (2\text{pts})$$

The general solution is

$$y = \underbrace{c_1 e^{x}}_{(1\text{pt})} + \underbrace{c_2 e^{-2x}}_{(1\text{pt})} + \underbrace{c_3 x e^{-2x}}_{(1\text{pt})} + \underbrace{(c_4 + c_5 x)\cos(3x) + (c_6 + c_7 x)\sin(3x)}_{(3\text{pts})}$$

9. **(12 points)** Find a basis for the solution space of the given homogeneous linear system.

$$x_1 - 3x_2 - 9x_3 - 5x_4 = 0$$
$$2x_1 + x_2 - 4x_3 + 11x_4 = 0$$
$$x_1 + 3x_2 + 3x_3 + 13x_4 = 0.$$

(1 pt)

$$\left[\begin{array}{cccc|c} 1 & -3 & -9 & -5 & 0 \\ 2 & 1 & -4 & 11 & 0 \\ 1 & 3 & 3 & 13 & 0 \end{array}\right] \xrightarrow[-R_1+R_3]{-2R_1+R_2} \left[\begin{array}{cccc|c} 1 & -3 & -9 & -5 & 0 \\ 0 & 7 & 14 & 21 & 0 \\ 0 & 6 & 12 & 18 & 0 \end{array}\right]$$

$$\xrightarrow[\frac{1}{6}R_1]{\frac{1}{7}R_2} \left[\begin{array}{cccc|c} 1 & -3 & -9 & -5 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{array}\right] \xrightarrow{-R_2+R_3} \left[\begin{array}{cccc|c} 1 & -3 & -9 & -5 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(3 pts)

(1 pt)

$$\Rightarrow \quad x_1 - 3x_2 - 9x_3 - 5x_4 = 0$$
$$x_2 + 2x_3 + 3x_4 = 0$$

x_3 and x_4 are free variables.

let $x_3 = t$, $x_4 = s$. (1 pt)

$$\Rightarrow x_2 = -2x_3 - 3x_4 = -2t - 3s \quad \text{(1 pt)}$$

$$x_1 = 3x_2 + 9x_3 + 5x_4 = -6t - 9s + 9t + 5s$$
$$= 3t - 4s \quad \text{(1 pt)}$$

The solution set $= \{(3t-4s, -2t-3s, t, s) \mid t, s \in \mathbb{R}\}$ (2 pts)

basis $= \{(3,-2,1,0), (-4,-3,0,1)\}$ (2 pts).

10. **(13 points)** Find a particular solution y_p of the differential equation

$$y'' + 2y' - 3y = 1 + xe^x.$$

Given that $y_c = c_1e^x + c_2e^{-3x}$ is the general solution of $y'' + 2y' - 3y = 0$.

Let $y_T = A + Be^x + cxe^x$

To remove the duplication with y_c, we adjust the form

to be $y_p = A + Bxe^x + cx^2e^x$ (3pts)

$\Rightarrow y_p' = Be^x + Bxe^x + 2cxe^x + cx^2e^x$

$= Be^x + (B+2C)xe^x + cx^2e^x$ (1pt)

$\Rightarrow y_p'' = Be^x + (B+2C)e^x + (B+2C)xe^x + 2Cxe^x + cx^2e^x$

$= (2B+2C)e^x + (B+4C)xe^x + cx^2e^x$ (2pts)

Substituting in the DE, we get

$(2B+2C)e^x + (B+4C)xe^x + \cancel{cx^2e^x} + 2Be^x + 2(B+2C)xe^x$

$+ \cancel{2Cx^2e^x} - 3A - 3Bxe^x - \cancel{3Cx^2e^x} = 1 + xe^x$ (3pts)

(1pt)

$\Rightarrow -3A = 1 \Rightarrow A = -\frac{1}{3}$, $4B + 2C = 0 \Rightarrow C = -2B$,

$8C = 1 \Rightarrow C = \frac{1}{8} \Rightarrow B = -\frac{1}{16}$.

(1pt) (1pt)

So, a particular solution y_p of the DE is

$y_p = -\frac{1}{3} - \frac{1}{16}xe^x + \frac{1}{8}x^2e^x$ (1pt)