1. The differential equation

$$x''' + 7x'' + 9x' - 2x = \sin 2t$$

can be converted to a system of linear differential equations of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \sin 2t \end{bmatrix}$$

where $A =$

(a) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -9 & -7 \end{bmatrix}$ (correct)

(b) $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 2 & -9 & -7 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ -9 & 2 & 7 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & -9 & -7 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 2 & 9 & 7 \end{bmatrix}$

2. If $A = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$, then $e^{At} =$

(a) $\begin{pmatrix} 1+2t & -t \\ 4t & 1-2t \end{pmatrix}$ (correct)

(b) $\begin{pmatrix} 1+2t & -t \\ 4t & -2t \end{pmatrix}$

(c) $\begin{pmatrix} 1+2t & 1-t \\ 1+4t & -2t \end{pmatrix}$

(d) $\begin{pmatrix} 1+2t & 2t \\ 4t & 1-2t \end{pmatrix}$

(e) $\begin{pmatrix} 2t & 1-t \\ 4t & 1-2t \end{pmatrix}$

3. Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 4 & 2 \end{bmatrix}$. If $A^{-1} = aI + bA + cA^2$, then $a + b + c =$

(a) 1 (correct)

(b) 0

(c) $\frac{1}{2}$

(d) $\frac{3}{4}$

(e) -2

4. Let $A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$ with $e^{At} = \begin{pmatrix} t+1 & -t \\ t & 1-t \end{pmatrix}$. If x is a solution of the IVP

$$x' = Ax + \begin{pmatrix} \frac{1}{t} \\ \frac{1}{t} \end{pmatrix};\ x(1) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

where $t > 0$, then $x(e) =$

(a) $\begin{pmatrix} 3e \\ 3e-3 \end{pmatrix}$ (correct)

(b) $\begin{pmatrix} 3e \\ 3e+4 \end{pmatrix}$

(c) $\begin{pmatrix} 3e \\ 3e-1 \end{pmatrix}$

(d) $\begin{pmatrix} 3e \\ e-4 \end{pmatrix}$

(e) $\begin{pmatrix} 3e \\ e+2 \end{pmatrix}$

5. If the solution of the first order linear differential equation
$y' - y = 5e^{-\frac{x}{2}}$; $y(0) = -2$ is given by $y = \dfrac{Ae^{x} - Be^{-\frac{x}{2}}}{3}$, then $2A + B =$

(a) 18 (correct)
(b) 16
(c) 24
(d) 12
(e) 10

6. If the differential equation

$$(2ye^{ay} + x^2)\,dx + (e^y + x(2+6y)e^{ay})\,dy = 0$$

is exact, then $a =$

(a) 3 (correct)
(b) 6
(c) 2
(d) 12
(e) 1

7. The form of a particular solution of

$$y'' - 2y' + y = -3e^x + 8xe^x + 10$$

is

(a) $(Ax^2 + Bx^3)e^x + k$ (correct)

(b) $(A + Bx)e^x + kx$

(c) $(Ax^2 + Bx^3)e^x + kx^2$

(d) $(Ax + Bx^2)e^x + k$

(e) $(A + Bx)e^x + kx^2$

8. The solution of $x' = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} x$, $x(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ at $t = \frac{\pi}{4}$ equals

(a) $\begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{\frac{\pi}{4}}$ (correct)

(b) $\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{\frac{\pi}{4}}$

(c) $\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{\frac{\pi}{4}}$

(d) $\begin{pmatrix} -2 \\ 0 \end{pmatrix} e^{\frac{\pi}{4}}$

(e) $\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-\frac{\pi}{4}}$

9. The characteristic equation of A is $(\lambda+1)(\lambda-5)^3 = 0$, where we have two linearly independent eigenvectors corresponding to $\lambda = 5$. The Jordan normal form of A is

(a) $\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{pmatrix}$ (correct)

(b) $\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{pmatrix}$

(c) $\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{pmatrix}$

(d) $\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$

(e) $\begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & 5 & 0 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{pmatrix}$

10. The rank of the matrix is $\begin{pmatrix} 1 & 3 & 2 & 4 \\ -1 & -3 & 5 & 3 \\ 2 & 6 & 5 & 9 \\ 3 & 9 & 6 & m \end{pmatrix}$ is 2 if $m =$

(a) 12 (correct)
(b) 10
(c) 8
(d) 6
(e) 4

11. The three vectors $v_1 = (0,3,0)$, $v_2 = (A,2,1)$ and $v_3 = (B,1,3)$ do not form a basis for $\mathbb{R}^3$ if

(a) $B = 3A$ (correct)

(b) $B = 2A$

(c) $B = A$

(d) $B = -2A$

(e) $B = 5A$

12. The general solution of $X' = \begin{bmatrix} 4 & -1 & -1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} x$ can be written as

$$X = c_1 \begin{bmatrix} a \\ 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ b \\ 0 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} 1 \\ 0 \\ c \end{bmatrix} e^{3t}.$$

Then $a - 2b + 2c =$

(a) 1 (correct)

(b) 0

(c) 2

(d) −1

(e) −2

13. **(11 points)** Use the method of variation of parameters to find the general solution of the differential equation

$$y'' - 2y' + y = \frac{e^x}{x^2+1}.$$

$y'' - 2y' + y = 0.$

The auxiliary equation is $m^2 - 2m + 1 = 0$

$\Rightarrow (m-1)^2 = 0 \Rightarrow m = 1, 1.$

$y_c = c_1 e^x + c_2 x e^x.$ (2pts)

Assume $y_p = u_1(x)\, e^x + u_2(x)\, x e^x$ (1pt)

$$W(e^x, xe^x) = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x} \quad \text{(1pt)}$$

$$W_1 = \begin{vmatrix} 0 & xe^x \\ \frac{e^x}{x^2+1} & e^x + xe^x \end{vmatrix} = \frac{-xe^{2x}}{x^2+1} \quad \text{(1pt)}$$

$$W_2 = \begin{vmatrix} e^x & 0 \\ e^x & \frac{e^x}{x^2+1} \end{vmatrix} = \frac{e^{2x}}{x^2+1} \quad \text{(1 pt)}$$

$$u_1(x) = \int \frac{W_1}{W}\, dx = -\int \frac{x}{x^2+1}\, dx = -\frac{1}{2}\ln(x^2+1) \quad \text{(2pts)}$$

$$u_2(x) = \int \frac{W_2}{W}\, dx = \int \frac{1}{x^2+1}\, dx = \tan^{-1} x \quad \text{(2pts)}$$

$$\Rightarrow y_p = -\frac{e^x}{2}\ln(x^2+1) + x e^x \tan^{-1} x$$

The general soln is: $y = c_1 e^x + c_2 x e^x - \frac{e^x}{2}\ln(x^2+1) + x e^x \tan^{-1} x$

(1pt)

14. **(11 points)** Let $A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$. Determine whether or not the given matrix A is diagonalizable. If it is, find a diagonalizing matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

$$\begin{vmatrix} 1-\lambda & 2 \\ 5 & 4-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(4-\lambda) - 10 = 0$$

(2pts)

$$\Rightarrow \lambda^2 - 5\lambda - 6 = 0 \Rightarrow (\lambda-6)(\lambda+1) = 0 \Rightarrow \lambda = 6, -1$$

For $\lambda = 6$: let $K = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \neq 0$ s.t $(A - 6I)K = 0 \Rightarrow$

$$\left[\begin{array}{cc|c} -5 & 2 & 0 \\ 5 & -2 & 0 \end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{cc|c} -5 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow -5k_1 + 2k_2 = 0$$

$$\Rightarrow k_2 = \frac{5}{2}k_1, \quad K_{\lambda=6} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$ (2pts)

For $\lambda = -1$: let $K = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \neq 0$ s.t $(A + I)K = 0 \Rightarrow$

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 5 & 5 & 0 \end{array}\right] \Rightarrow k_1 = -k_2. \quad K_{\lambda=-1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$ (2pts)

Since order$(A) = 2$ and it has two Linearly independent eigenvectors $\Rightarrow$ A is diagonalizable (1pt)

$$D = \begin{bmatrix} 6 & 0 \\ 0 & -1 \end{bmatrix}$$ (2pts)

$$P = \begin{bmatrix} 2 & 1 \\ 5 & -1 \end{bmatrix}$$ (2pts)

15. **(11 points)** Find the general solution of $X' = AX$ where $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

$$\begin{vmatrix} 1-\lambda & 2 & -1 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^3 = 0 \Rightarrow \lambda = 1, 1, 1$$
(1pt)

let $K = \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix}$ s.t $(A - I)K = 0 \Rightarrow$

$$\left[\begin{array}{ccc|c} 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow 2K_2 = K_3, \; K_1 \text{ is free.}$$
(1pt) (1pt)

$$K = \begin{bmatrix} K_1 \\ K_2 \\ 2K_2 \end{bmatrix} = K_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + K_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

$$\therefore \; X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t, \quad X_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} e^t$$
(1pt) (1pt)

To find X_3 we find (nonzero) vectors V_1, V_2 s.t

$(A - I)^2 V_2 = 0$ and $(A - \lambda I)V_2 = V_1$.

$$\Rightarrow \begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^2 V_2 = 0 \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} V_2 = 0$$

let $V_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. (2pts) $\Rightarrow V_1 = \begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$. (1pt)

$$X_3 = (V_1 t + V_2)e^t = \begin{bmatrix} -te^t \\ 0 \\ e^t \end{bmatrix}.$$
(2pts)

The general solution is $X = c_1 X_1 + c_2 X_2 + c_3 X_3$ (1pt)