1. Which one of the following differential equations is **not** separable?

(a) $x\dfrac{dy}{dx} + 2y = 2xy^2$ $\Rightarrow \dfrac{dy}{dx} = \dfrac{2xy^2 - 2y}{x} = \dfrac{2y(xy-1)}{x}$ not sep. (correct)

(b) $\dfrac{dy}{dx} - \sqrt{x}y = 0$ $\Rightarrow \dfrac{dy}{dx} = \sqrt{x}\cdot y$ ✓

(c) $\dfrac{dy}{dx} = 5e^{x-4y}$ $= 5e^{x}\cdot e^{-4y}$ ✓

(d) $x\dfrac{dy}{dx} - y = 2x^2y$ $\Rightarrow x\dfrac{dy}{dx} = y + 2x^2y = y(1+2x^2) \Rightarrow \dfrac{dy}{dx} = y\cdot\dfrac{1+2x^2}{x}$ ✓

(e) $\dfrac{dy}{dx} = \dfrac{x+1}{y-1}$ $= (x+1)\cdot\dfrac{1}{y-1}$ ✓

2. Find all values of $a$ such that $y = ax - \ln x$ is a solution of the differential equation

$$x^2y'' + xy' - y = \ln x.$$

$y' = a - \frac{1}{x}$

$y'' = \frac{1}{x^2}$

Sub. in the ODE:

(a) $a \in (-\infty, \infty)$ (correct)

(b) $a \in (0, \infty)$

(c) $a \in (-\infty, 0)$

(d) $a \in [1, \infty)$

(e) $a \in [-1, \infty)$

$LHS = x^2y'' + xy' - y = x^2\left(\frac{1}{x^2}\right) + x\left(a - \frac{1}{x}\right) - (ax - \ln x)$

$= 1 + ax - 1 - ax + \ln x$

$= \ln x$ for all $a \in (-\infty, \infty)$

$= RHS$

3. If the system

$$\begin{aligned} 4x + 3y &= 5 \\ 8x + ky &= 10 \end{aligned}$$

has infinitely many solutions, then $k =$

(a) 6 (correct)

(b) 3

(c) 4

(d) $-3$

(e) $-4$

$$\left[\begin{array}{cc|c} 4 & 3 & 5 \\ 8 & k & 10 \end{array}\right] \xrightarrow{R_2 \to -2R_1 + R_2} \left[\begin{array}{cc|c} 4 & 3 & 5 \\ 0 & k-6 & 0 \end{array}\right]$$

↳ if $k-6=0$, i.e., $k=6$, then there are infinitely many sol.

[ $y$ is a free variable ].

4. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 1 \end{bmatrix}.$$

If

$$A^{-1} = \begin{bmatrix} 1 & a & b \\ 0 & c & d \\ 0 & 0 & 1 \end{bmatrix},$$

then $a + b + c + d =$

(a) $-2$ (correct)

(b) $-\frac{3}{4}$

(c) $-\frac{1}{2}$

(d) $\frac{1}{4}$

(e) $\frac{5}{4}$

$$\begin{array}{cc} A & I \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 4 & 5 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to \frac{1}{4}R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5/4 & 0 & 1/4 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to -2R_2 + R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & 1 & -1/2 & 0 \\ 0 & 1 & 5/4 & 0 & 1/4 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to -\frac{1}{2}R_3 + R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1/2 & -1/2 \\ 0 & 1 & 5/4 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to -\frac{5}{4}R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1/2 & -1/2 \\ 0 & 1 & 0 & 0 & 1/4 & -5/4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{cc} I & A^{-1} \end{array}$$

$a = -\frac{1}{2}$, $b = -\frac{1}{2}$, $c = \frac{1}{4}$, $d = -\frac{5}{4}$

$a+b+c+d = -2$

5. If the differential equation

$$\underbrace{(Ax^2y + (A-B)y\sin(2x))}_{M}\,dx + \underbrace{(x^3 + B\sin^2 x)}_{N}\,dy = 0$$

is **exact**, then

$M_y = Ax^2 + (A-B)\sin(2x)$

$N_x = 3x^2 + B \cdot 2\sin x\cos x$

$= 3x^2 + B\sin(2x)$

$M_y = N_x \Rightarrow A = 3$ & $A - B = B$

$\Rightarrow A = 3$ & $3 - B = B$

$\Rightarrow A = 3$ & $B = \frac{3}{2}$

(a) $A = 3$, $B = 3/2$ (correct)

(b) $A = 2$, $B = 2/3$

(c) $A = 3$, $B = -3/2$

(d) $A = 1$, $B = -1$

(e) $A = -1$, $B = -2$

6. Find the **determinant** of the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

$$\det(A) = 2\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix}$$

$= 2\left[2(3) - (2) + 0\right] - \left[1(3) - 1(0) + 0\right]$

$= 2(4) - 3 = 8 - 3 = 5$

(a) 5 (correct)

(b) $-4$

(c) 0

(d) 8

(e) $-3$

7. A one-parameter family of solutions of the differential equation

$$\frac{dy}{dx} = 2xy + 2x + y + 1 = 2x(y+1) + (y+1) = (y+1)(2x+1) \quad \text{sep.}$$

is

$$\Rightarrow \frac{1}{y+1}\,dy = (2x+1)\,dx$$

$$\Rightarrow \ln|y+1| = x^2 + x + c_1$$

$$\Rightarrow |y+1| = e^{x^2+x+c_1}$$

$$\Rightarrow y+1 = \underbrace{\pm e^{c_1}}_{c} \cdot e^{x^2+x}$$

$$\Rightarrow y = -1 + c e^{x^2+x}.$$

(a) $y = -1 + ce^{x^2+x}$ (correct)

(b) $y = 1 + ce^{x^2+x}$

(c) $y = -2 + ce^{2x^2+x}$

(d) $y = -1 + ce^{x^2-2x}$

(e) $y = 1 + ce^{x^2-x}$

8. If a ~~particular~~ particle moving in a straight line has ~~a~~ acceleration

$$a(t) = \frac{3}{(t+1)^3},$$

an initial position $x(0) = 0$, and an initial velocity $v(0) = 0$, then $x(2) =$ (the position of the particle at $t = 2$)

$$a(t) = 3(t+1)^{-3}$$

$$v(t) = \int 3(t+1)^{-3}\,dt = -\frac{3}{2}(t+1)^{-2} + C$$

$$v(0) = 0 \Rightarrow 0 = -\frac{3}{2} + C \Rightarrow C = \frac{3}{2}$$

$$v(t) = -\frac{3}{2}(t+1)^{-2} + \frac{3}{2}$$

$$\Rightarrow x(t) = \int -\frac{3}{2}(t+1)^{-2} + \frac{3}{2}\,dt$$

$$= \frac{3}{2}(t+1)^{-1} + \frac{3}{2}t + D$$

$$x(0) = 0 \Rightarrow 0 = \frac{3}{2} + 0 + D \Rightarrow D = -\frac{3}{2}$$

$$\text{So } x(t) = \frac{3}{2}(t+1)^{-1} + \frac{3}{2}t - \frac{3}{2}$$

$$x(2) = \frac{3}{2}(3)^{-1} + \frac{3}{2}(2) - \frac{3}{2}$$

$$= \frac{1}{2} + 3 - \frac{3}{2} = 3 - 1 = 2$$

(a) 2 (correct)

(b) $\frac{3}{2}$

(c) 4

(d) $\frac{10}{3}$

(e) $\frac{9}{4}$

9. The population of a community is known to increase at a rate proportional to the number of people present at time $t$. If an initial population $P_0$ has tripled in 8 years, how long will it take to quadruple?

(a) $\dfrac{8\ln 4}{\ln 3}$ (correct)

(b) $\dfrac{4\ln 4}{\ln 3}$

(c) $\dfrac{3\ln 3}{\ln 2}$

(d) $\dfrac{2\ln 4}{\ln 3}$

(e) $\dfrac{8\ln 3}{\ln 2}$

• $P(t) = Ce^{kt}$

. $P(0) = P_0 \Rightarrow P_0 = Ce^{k(0)} \Rightarrow \boxed{C = P_0}$

. $P(8) = 3P_0 \Rightarrow 3P_0 = P_0 e^{8k} \Rightarrow e^{8k} = 3$

$\Rightarrow 8k = \ln 3 \Rightarrow \boxed{k = \frac{\ln 3}{8}}$

. $P(t) = 4P_0 \Rightarrow 4P_0 = P_0 e^{kt}$

$\Rightarrow 4 = e^{kt}$

$\Rightarrow \ln 4 = kt$

$\Rightarrow t = \dfrac{\ln 4}{k} = \dfrac{8\ln 4}{\ln 3}$

10. When reducing the matrix

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{bmatrix}$$

into an **echelon form**, we obtain

(a) $\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ (correct)

(b) $\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 6 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

$$\xrightarrow[R_3 \to -2R_1+R_3]{R_2 \to -3R_1+R_2} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 2 & 4 & 8 \\ 0 & 3 & 7 & 15 \end{bmatrix}$$

$$\xrightarrow{R_2 \to \frac{1}{2}R_2} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 3 & 7 & 15 \end{bmatrix}$$

$$\xrightarrow{R_3 \to -3R_2+R_3} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

(e) $$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

11. **(15 points)** Solve the initial-value problem:

$$\frac{dy}{dx} - \frac{2xy}{x^2+1} = x^2+2, \; y(0) = 1.$$

• Linear in y and in standard form : $P(x) = -\frac{2x}{x^2+1}$ , $g(x) = x^2+2$

Both conts on $(-\infty,\infty)$.

∴ an integrating factor : $\mu(x) = e^{\int P(x)\,dx}$ ② $= e^{-\int \frac{2x}{x^2+1}dx}$ ② $= e^{-\ln(x^2+1)}$

$= e^{\ln(x^2+1)^{-1}} = (x^2+1)^{-1} = \frac{1}{x^2+1}$ . ②

. Multiply the ODE by $\mu(x)$:

$$\frac{1}{x^2+1}\frac{dy}{dx} - \frac{2x}{(x^2+1)^2}y = \frac{x^2+2}{x^2+1}$$ ①

$$\Rightarrow \frac{d}{dx}\left[\frac{1}{x^2+1}\cdot y\right] = \frac{x^2+2}{x^2+1}$$ ①

$$\Longrightarrow \frac{1}{x^2+4}\cdot y = \int \frac{x^2+2}{x^2+1}dx = \int \left(1+\frac{1}{x^2+1}\right)dx = x + \tan^{-1}x + C$$

① ②

$$\Longrightarrow y = (x^2+1)(x+\tan^{-1}x + C)$$ ①

Find c using $y(0)=1$

$$1 = (0+1)(0+0+C) \Longrightarrow C = 1$$ ②

The solution of the IVP is

$$y = (x^2+1)(x+\tan^{-1}x+1).$$ ①

12. **(15 points)** Verify that the following differential equation is exact, and then solve it:

$$\left(\frac{1}{x^3} + ye^x\right) dx + (e^x + \cot y)\, dy = 0.$$

. $M(x,y) = \frac{1}{x^3} + ye^x \Rightarrow M_y(x,y) = 0 + e^x = e^x$ ②

. $N(x,y) = e^x + \cot y \Rightarrow N_x(x,y) = e^x + 0 = e^x$ ②

Since $M_y(x,y) = N_x(x,y)$ then the DE is exact. ①

Then there is a function $f(x,y)$ such that

$$f_x(x,y) = M(x,y) \text{ and } f_y(x,y) = N(x,y)$$ ②

. $f_x(x,y) = M(x,y) = \frac{1}{x^3} + ye^x \Rightarrow f(x,y) = \int (x^{-3} + ye^x)\, dx$

$$= \frac{x^{-2}}{-2} + ye^x + g(y)$$ ②

$$\Rightarrow f_y(x,y) = 0 + e^x + g'(y)$$

$$\Rightarrow e^x + \cot y = e^x + g'(y)$$ ②

$$\Rightarrow g'(y) = \cot y$$

$$\Rightarrow g(y) = \int \cot y\, dy = \ln|\sin y|$$ ②

. Then

$$f(x,y) = -\frac{1}{2x^2} + ye^x + \ln|\sin y|$$

and the solution of the ODE is

$$f(x,y) = C$$

$$\Rightarrow \quad -\frac{1}{2x^2} + ye^x + \ln|\sin y| = C$$ ②

13. **(10 points)** Consider the system

$$\begin{aligned} x + y + z &= 1 \\ x - y + 2z &= -2 \\ 2x + 2y - z &= 5. \end{aligned}$$

Use Cramer's Rule to find the value of $z$.
(**Note: You will get zero if you use another method**).

- $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ ① $\Rightarrow \det(A) = \begin{vmatrix} \overset{\oplus}{1} & \overset{\ominus}{1} & \overset{\oplus}{1} \\ 1 & -1 & 2 \\ 2 & 2 & -1 \end{vmatrix}$

$= 1 \begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} + 1 \begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix}$

$= 1\,(1-4) - 1\,(-1-4) + 1\,(2+2)$

$= -3 + 5 + 4 = 6$ ②

- $A_z = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -2 \\ 2 & 2 & 5 \end{bmatrix}$ ① $\Rightarrow \det(A_z) = \begin{vmatrix} \overset{\oplus}{1} & \overset{\ominus}{1} & \overset{\oplus}{1} \\ 1 & -1 & -2 \\ 2 & 2 & 5 \end{vmatrix}$

$= 1 \begin{vmatrix} -1 & -2 \\ 2 & 5 \end{vmatrix} - 1 \begin{vmatrix} 1 & -2 \\ 2 & 5 \end{vmatrix} + 1 \begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix}$

$= 1\,(-5+4) - 1\,(5+4) + 1\,(2+2)$

$= -1 - 9 + 4 = -6$ ②

- $z = \dfrac{\det(A_z)}{\det(A)}$ ②

$= \dfrac{-6}{6} = -1.$ ②

**Answer KEY**

| Q | MASTER | CODE01 | CODE02 | CODE03 | CODE04 |
|---|---|---|---|---|---|
| 1 | A | A | D | E | B |
| 2 | A | B | C | B | E |
| 3 | A | E | B | E | B |
| 4 | A | E | B | D | D |
| 5 | A | A | C | B | B |
| 6 | A | D | E | D | C |
| 7 | A | E | C | B | A |
| 8 | A | B | A | C | B |
| 9 | A | C | D | D | B |
| 10 | A | B | D | A | D |