1. If $y_p = Ax + B$ is a particular solution of the differential equation

$$y'' - y' - 2y = 2x,$$

then $A + B =$

(a) $-\dfrac{1}{2}$ (correct)

(b) $\dfrac{1}{2}$

(c) $\dfrac{3}{2}$

(d) 2

(e) 0

$y_p' = A$, $y_p'' = 0$

$y_p'' - y_p' - 2y_p = 2x$

$0 - A - 2(Ax+B) = 2x$

$-2Ax - A - 2B = 2x$

$\Rightarrow \begin{cases} -2A = 2 \\ -A - 2B = 0 \end{cases} \Rightarrow A = -1,\ B = +\frac{1}{2}$

$A + B = -1 + \frac{1}{2} = -\frac{1}{2}$

2. The general solution of the ODE

$$2y''' - 3y'' - 2y' = 0$$

is

(a) $y = c_1 + c_2 e^{2x} + c_3 e^{-x/2}$ (correct)

(b) $y = c_1 + c_2 e^{-2x} + c_3 e^{x/2}$

(c) $y = c_1 + c_2 e^{-x} + c_3 e^{-2x}$

(d) $y = c_1 e^{2x} + c_2 e^{-x/2}$

(e) $y = c_1 + c_2 e^{2x} + c_3 x e^{2x}$

$2r^3 - 3r^2 - 2r = 0$

$r(2r^2 - 3r - 2) = 0$

$r(2r+1)(r-2) = 0$

$r = 0, -\frac{1}{2}, 2$

$y = c_1 e^{0x} + c_2 e^{2x} + c_3 e^{-\frac{1}{2}x}$

$= c_1 + c_2 e^{2x} + c_3 e^{-x/2}$

3. Which set of the following functions is linearly independent on $(-\infty, \infty)$.

(a) $f(x) = 2,\ g(x) = x,\ h(x) = x^2$ no one is a linear combination of the others (correct)

(b) $f(x) = 1,\ g(x) = 2 - x^2,\ h(x) = 3 + x^2$ $\to f(x) = \frac{1}{5}g(x) + \frac{1}{5}h(x)$

(c) $f(x) = 5,\ g(x) = \sin^2 x,\ h(x) = \cos(2x)$

(d) $f(x) = 0,\ g(x) = \cos x,\ h(x) = e^{-x}$ $f(x) = 0 \cdot g(x) + 0 \cdot h(x)$

(e) $f(x) = x,\ g(x) = \sin(2x),\ h(x) = \sin x \cos x$ $g(x) = 2h(x) + 0 \cdot f(x)$

$h(x) = \frac{1}{5}f(x) - 2g(x)$

4. An appropriate form of a particular solution y_p for the differential equation

$$y^{(5)} - y^{(3)} = -3x^2 + 5 + 2e^x$$

is

(a) $y_p = Ax^3 + Bx^4 + Cx^5 + Dxe^x$ (correct)

(b) $y_p = A + Bx + Cx^2 + De^x$

(c) $y_p = A + Bx + Cx^2 + Dxe^x$

(d) $y_p = Ax + Bx^2 + Cx^3 + Dxe^x$

(e) $y_p = Ax^2 + Bx^3 + Cx^4 + Dx^2e^x$

$y^{(5)} - y^{(3)} = 0 \Rightarrow r^5 - r^3 = 0 \Rightarrow r^3(r^2-1) = 0 \Rightarrow r = 0$ (3 times), $r = -1$, $r = 1$

$y_c = c_1 + c_2x + c_3x^2 + c_4e^{-x} + c_5e^x$

there is a duplicate with y_c

$g(x) = -3x^2 + 5 + 2e^x \Rightarrow y_p = \underbrace{A + Bx + Cx^2}_{\times x^3} + \underbrace{De^x}_{\times x}$

$\Rightarrow y_p = Ax^3 + Bx^4 + Cx^5 + Dxe^x$, no duplicate with y_c

5. A linear homogeneous differential equation whose general solution is

$$y = (A + Bx + Cx^2)\, e^{2x}$$

is given by

(a) $y''' - 6y'' + 12y' - 8y = 0$ (correct)

(b) $y''' - 6y'' + 10y' - 8y = 0$

(c) $y'' - 6y' - 8y = 0$

(d) $y''' + 6y'' + 6y' + 8y = 0$

(e) $y''' - 8y = 0$

$r = 2$ repeated 3 times

$(r-2)^3 = 0$

$r^3 - 3r^2(2) + 3r \cdot 2^2 - 2^3 = 0$

$r^3 - 6r^2 + 12r - 8 = 0$

$y''' - 6y'' + 12y' - 8y = 0$

6. Let A be a 5×8 matrix with real entries. If the rank of A is 3, then the dimension of the solution space of the system $A\mathbf{X} = 0$ is

(a) 5 (correct)

(b) 4

(c) 3

(d) 2

(e) 8

$\underbrace{\#\text{ of Leading Var.}}_{= \text{rank of } A \, = 3} + \underbrace{\#\text{ of Free Var.}}_{= \text{dim of Sol. space}} = 8$

So dim of Sol. space $= 8 - 3 = 5$

7. Let $u = (1,1,0)$, $v = (0,1,1)$, $w = (1,2,-2)$ be vectors in $\mathbb{R}^3$. If

$$(2,3,4) = au + bv + cw,$$

then $a^2 + b^2 + c^2 =$

$$\begin{cases} a + c = 2 \\ a + b + 2c = 3 \\ b - 2c = 4 \end{cases}$$

(a) 14
(b) 12
(c) 9
(d) 16
(e) 20

Solving The system, we get

$a = 3,\ b = 2,\ c = -1$

So $a^2 + b^2 + c^2 = 9 + 4 + 1 = 14$

(correct)

8. Find **all** real values of k so that the vectors

$$u = (1,2,4),\ v = (2,-1,k),\ w = (-1,4,2)$$

form a basis for $\mathbb{R}^3$. The three vectors form a basis for $\mathbb{R}^3$ if and only if they L. indep & they are L. indep if and only

(a) $k \neq 3$
(b) $k \neq 0$
(c) $k \neq 1$
(d) $k \neq 2$
(e) $k \neq -5$

$$\begin{array}{c} \quad u \quad\ v \quad\ w \\ \begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 4 \\ 4 & k & 2 \end{vmatrix} \neq 0 \end{array}$$

(correct)

$1(-2-4k) - 2(4-16) - (2k+4) \neq 0$

$-2 - 4k + 24 - 2k - 4 \neq 0$

$\Rightarrow \quad -6k + 18 \neq 0$

$\Rightarrow \quad k \neq 3$

9. The dimension of the subspace

$$W = \{(x, y, z, w) : x - y + 2z - w = 0\}$$

of $\mathbb{R}^4$ is

= Solution space of the System

$x - y + 2z - w = 0$

$[1 \ \ -1 \ \ 2 \ \ -1 \ \vdots \ 0]$

Lead Var: x

Free Var: $y, z, w \Rightarrow \dim(W) = 3$

(a) 3 (correct)

(b) 2

(c) 1

(d) 4

(e) 0

10. Which one of the following statements is **TRUE** about the following subsets of $\mathbb{R}^3$

$U = \{(x, y, z) : x + y = z\}$ ✓ check the two conditions ✓

$V = \{(x, y, z) : x + y = z + 1\}$ $(0,0,0) \notin V$. So V is not a subspace

$W = \{(x, y, z) : x^2 + y^2 = z^2\}$ $(1,0,1), (0,1,1) \in W$

But $(1,0,1) + (0,1,1) = (1,1,2) \notin W$

So W is not a subspace

(a) U is a subspace of $\mathbb{R}^3$ (correct)

(b) V is a subspace of $\mathbb{R}^3$

(c) W is a subspace of $\mathbb{R}^3$

(d) U and W are subspaces of $\mathbb{R}^3$

(e) none of U, V, W is a subspace of $\mathbb{R}^3$

11. **(10 points)** Solve $2y''' + 5y'' + 3y' + 2y = 0$.

Char. eq: $2\cancel{m}^3 + 5r^2 + 3r + 2 = 0$ (2)

By inspection $r = -2$ is a root $\left(2(-2)^3 + 5(-2)^2 + 3(-2) + 2 = -16 + 20 - 6 + 2 = 0\right.$
(2)

So $r+2$ is a factor of $2r^3 + 5r^2 + 3r + 2$

By long division, we get

$$\begin{array}{r|l} & 2r^2 + r + 1 \\ \hline r+2 & 2r^3 + 5r^2 + 3r + 2 \\ & \underline{2r^3 + 4r^2} \\ & \quad r^2 + 3r + 2 \\ & \quad \underline{r^2 + 2r} \\ & \qquad r + 2 \\ & \qquad \underline{r + 2} \\ & \qquad 0 \end{array}$$

(2) $(r+2)(2r^2 + r + 1) = 0$

$\Rightarrow r = -2, \quad r = \dfrac{-1 \pm \sqrt{1 - 4(2)(1)}}{2(2)}$

$\Rightarrow r = -2, \quad r = -\frac{1}{4} \pm \frac{1}{4}\sqrt{7}\, i$
(1)

The Sol is

$$y = c_1 e^{-2x} + c_2 e^{-\frac{x}{4}} \cos\left(\frac{\sqrt{7}}{4}x\right) + c_3 e^{-x/4} \sin\left(\frac{\sqrt{7}}{4}x\right).$$

(1) (1) (1)

12. **(15 points)** Solve $y'' + y = 4\cos x$ by the Method of undetermined coefficients.

• y_c: $y'' + y = 0$: Char. eq: $r^2+1=0 \Rightarrow r = \pm i$ (2)

$$\Rightarrow \boxed{y_c = c_1 \cos x + c_2 \sin x}$$ (2)

• y_p: $g(x) = 4\cos x$

Since there is a duplication between $g(x)$ and y_c, then

(3) $\boxed{y_p = Ax\cos x + Bx\sin x}$ Find A, B?

(1) $y_p' = -Ax\sin x + A\cos x + Bx\cos x + B\sin x$

(2) $y_p'' = -Ax\cos x - A\sin x - A\sin x - Bx\sin x + B\cos x + B\cos x$

$= -Ax\cos x - Bx\sin x - 2A\sin x + 2B\cos x$

Sub. in the ODE:

$$y_p'' + y_p = 4\cos x$$

$$\cancel{-Ax\cos x} - \cancel{Bx\sin x} - 2A\sin x + 2B\cos x + \cancel{Ax\cos x} + \cancel{Bx\sin x} = 4\cos x$$

$\Rightarrow -2A\sin x + 2B\cos x = 4\cos x$

$\Rightarrow -2A = 0$ & $2B = 4$

$\Rightarrow A = 0$ & $B = 2$ (1) + (2)

So $\boxed{y_p = 2x\sin x}$

• G.S.: $y = y_c + y_p$ (2)

$= c_1\cos x + c_2\sin x + 2x\sin x$

13. **(15 points)** Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 & 3 \\ -1 & 1 & 2 & 3 & 0 \\ 0 & 2 & 1 & 2 & 0 \\ -1 & 3 & 3 & 5 & 0 \end{bmatrix}$$

(a) Find a basis for the column space of A.

(b) Find the column rank of A.

(c) Find the rank of A.

a)

• Echelon form of A

$$A \xrightarrow[R_4 \to R_1+R_4]{R_2 \to R_1+R_2} \begin{bmatrix} 1 & 0 & -1 & 0 & 3 \\ 0 & 1 & 1 & 3 & 3 \\ 0 & 2 & 1 & 2 & 0 \\ 0 & 3 & 2 & 5 & 3 \end{bmatrix} \quad 2$$

$$\xrightarrow[R_4 \to -3R_2+R_4]{R_3 \to -2R_2+R_3} \begin{bmatrix} 1 & 0 & -1 & 0 & 3 \\ 0 & 1 & 1 & 3 & 3 \\ 0 & 0 & -1 & -4 & -6 \\ 0 & 0 & -1 & -4 & -6 \end{bmatrix} \quad 2$$

$$\xrightarrow{R_3 \to -R_3} \begin{bmatrix} 1 & 0 & -1 & 0 & 3 \\ 0 & 1 & 1 & 3 & 3 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & -1 & -4 & -6 \end{bmatrix} \quad 1$$

$$\xrightarrow{R_4 \to R_3+R_4} \begin{bmatrix} 1 & 0 & -1 & 0 & 3 \\ 0 & 1 & 1 & 3 & 3 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = E \quad 1$$

- Pivot columns of E : 1^{st}, 2^{nd}, 3^{rd} columns
- A basis for Col(A) = { 1^{st}, 2^{nd}, 3^{rd} columns of A }

$$= \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \\ 3 \end{bmatrix} \right\}$$

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or $= \{ (1,-1,0,-1), (0,1,2,3), (-1,2,1,3). \}$

b) Column rank of A is $\underline{\underline{3}}$, since the basis of the column rank of A contains 3 vectors.

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c) rank of A = Column rank of A = 3

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