Answer KEY

Q	MASTER	CODE01	CODE02	CODE03	CODE04
1	A	B $_{2}$	A $_{1}$	A $_{1}$	D $_{3}$
2	A	C $_{1}$	D $_{3}$	B $_{5}$	A $_{1}$
3	A	C $_{3}$	D $_{5}$	C $_{3}$	E $_{5}$
4	A	A $_{5}$	C $_{4}$	D $_{2}$	E $_{4}$
5	A	D $_{4}$	C $_{2}$	D $_{4}$	D $_{2}$
6	A	E $_{9}$	A $_{6}$	B $_{13}$	B $_{13}$
7	A	D $_{12}$	D $_{10}$	E $_{12}$	A $_{12}$
8	A	C $_{13}$	E $_{7}$	D $_{11}$	B $_{11}$
9	A	E $_{7}$	E $_{8}$	B $_{7}$	C $_{6}$
10	A	C $_{10}$	A $_{9}$	E $_{8}$	E $_{10}$
11	A	B $_{8}$	B $_{13}$	E $_{6}$	B $_{8}$
12	A	E $_{6}$	B $_{12}$	E $_{9}$	B $_{9}$
13	A	E $_{11}$	B $_{11}$	C $_{10}$	B $_{7}$
14	A	B $_{15}$	B $_{16}$	D $_{20}$	B $_{20}$
15	A	A $_{14}$	C $_{20}$	A $_{15}$	A $_{18}$
16	A	D $_{20}$	D $_{15}$	C $_{16}$	A $_{14}$
17	A	A $_{18}$	D $_{17}$	C $_{14}$	E $_{16}$
18	A	A $_{16}$	A $_{14}$	B $_{17}$	D $_{17}$
19	A	B $_{19}$	E $_{18}$	C $_{18}$	A $_{15}$
20	A	C $_{17}$	A $_{19}$	C $_{19}$	A $_{19}$
21	A	A $_{24}$	B $_{24}$	B $_{23}$	A $_{25}$
22	A	A $_{23}$	A $_{23}$	A $_{22}$	A $_{23}$
23	A	B $_{25}$	A $_{22}$	A $_{25}$	B $_{22}$
24	A	A $_{21}$	A $_{21}$	A $_{21}$	B $_{21}$
25	A	A $_{22}$	B $_{25}$	B $_{24}$	B $_{24}$

Detailed Solution $\longrightarrow$

1. The general solution of the differential equation

$$\frac{dy}{dx} = 5e^{x-4y}$$

is

$e^{4y}\, dy = 5 e^{x}\, dx$

$\frac{1}{4} e^{4y} = 5 e^{x} + K$

(a) $y = \frac{1}{4} \ln\left(20e^{x} + c\right)$ ______ (correct)

(b) $y = 4 \ln\left(20e^{x} + c\right)$

(c) $y = \frac{1}{4} \ln\left(5e^{x} + c\right)$

(d) $y = 2 \ln\left(5e^{x} + c\right)$

(e) $y = \ln\left(10e^{x} + c\right)$

$e^{4y} = 20 e^{x} + 4K$

$4y = \ln(20 e^{x} + C)$, $C = 4K$

$y = \frac{1}{4} \ln(20 e^{x} + C)$

2. The general solution of the differential equation

$$\frac{dy}{dx} + 3x^2 y = 6x^2$$

is

$\mu(x) = e^{\int 3x^2 dx} = e^{x^3}$

$e^{x^3} \cdot y' + 3x^2 e^{x^3} y = 6x^2 e^{x^3}$

$\frac{d}{dx}[e^{x^3} \cdot y] = 6x^2 e^{x^3}$

(a) $y = 2 + ce^{-x^3}$ ______ (correct)

(b) $y = 3 + ce^{-x^3}$

(c) $y = 2 + cxe^{-x^2}$

(d) $y = cx - e^{-x^3}$

(e) $y = -1 + ce^{-x^3}$

$e^{x^3} \cdot y = 2 e^{x^3} + C$

$y = 2 + C e^{-x^3}$

3. The solution of the initial-value problem

$$-\frac{y}{x^2}dx + \left(\frac{1}{x}+1\right)dy = 0,\ y(1)=1$$

is

(a) $y = \dfrac{2x}{x+1}$ ________________ (correct)

(b) $y = \dfrac{x^2+1}{x+1}$

(c) $y = \dfrac{2x-1}{x}$

(d) $y = \dfrac{2x}{x^2+1}$

(e) $y = \dfrac{2}{x^2} - 1$

$\left.\begin{array}{l} M = -\frac{y}{x^2} \Rightarrow M_y = -\frac{1}{x^2} \\ N = \frac{1}{x}+1 \Rightarrow N_x = -\frac{1}{x^2} \end{array}\right] =,\ \text{exact}$

$F_x = M,\quad F_y = N$

$\Downarrow$

$F = \int -\frac{y}{x^2}\,dx = \frac{y}{x} + g(y)$

$F_y = \frac{1}{x} + g'(y) = N = \frac{1}{x}+1$

$\Rightarrow g'(y) = 1 \Rightarrow g(y) = y$

So $F(x,y) = \frac{y}{x} + y$

Sol : $\frac{y}{x} + y = c \overset{\text{IVP}}{\Longrightarrow} 1+1 = c \Rightarrow c=2$

$\frac{y}{x} + y = 2 \Rightarrow y\left(\frac{1}{x}+1\right) = 2$

$\Rightarrow y = \frac{2x}{x+1}$

4. If $(x,y,z) = (a,b,c)$ is the solution of the system

$$\begin{cases} x+3y+2z=5 \\ 2x+7y+5z=8 \\ 2x+3y+5z=5 \end{cases}$$

then $a+b+c=$

$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 2 & 7 & 5 & 8 \\ 2 & 3 & 5 & 5 \end{array}\right] \xrightarrow[R_3 \to -2R_1+R_3]{R_2 \to -2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & 1 & 1 & -2 \\ 0 & -3 & 1 & -5 \end{array}\right]$

$\xrightarrow{R_3 \to 3R_2+R_3} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 4 & -11 \end{array}\right]$

(a) $\dfrac{25}{4}$ ________________ (correct)

(b) $\dfrac{19}{4}$

(c) $\dfrac{33}{4}$

(d) 4

(e) $\dfrac{9}{2}$

$z = -\frac{11}{4}$

$y = -2 - z = -2 + \frac{11}{4} = \frac{3}{4}$

$x = 5 - 3y - 2z = 5 - \frac{9}{4} + \frac{22}{4} = \frac{20-9+22}{4} = \frac{33}{4}$

$a+b+c = \frac{33}{4} + \frac{3}{4} - \frac{11}{4} = \frac{36-11}{4} = \frac{25}{4}$

5. A basis for the solution space of the system

$$\begin{cases} x_1 - 2x_2 + 3x_3 = 0 \\ 2x_1 - 3x_2 - 4x_3 = 0 \end{cases}$$

$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 2 & -3 & -4 & 0 \end{array}\right]$

is

$\xrightarrow{R_2 \to -2R_1 + R_2} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 1 & -10 & 0 \end{array}\right]$ L.V.: x_1, x_2 F.V.: $x_3 = \alpha$

$x_2 = 10\alpha$

$x_1 = 2x_2 - 3\alpha = 20\alpha - 3\alpha = 17\alpha$

(a) $\{(17, 10, 1)\}$ (correct)

(b) $\{(1, -2, 3)\}$

(c) $\{(2, -3, -4)\}$

(d) $\{(3, -5, -1), (0, 1, 17)\}$

(e) $\{(11, 7, 1), (1, -2, 3)\}$

$(x_1, x_2, x_3) = (17\alpha, 10\alpha, \alpha)$

$= (17, 10, 1)\alpha, \quad \alpha \in \mathbb{R}$

6. If $y(x)$ is the solution of the initial-value problem

$$y'' - y' - 2y = 4e^x, \; y(0) = 3, \; y'(0) = -1$$

then $y(1) =$

Sol: $y = 3e^{-x} + 2e^{2x} - 2e^{x}$

(a) $3e^{-1} + 2e^2 - 2e$ (correct)

(b) $2e^{-1} - 3e^2 - 2e$

(c) $e^{-1} + 2e^2 + 3e$

(d) $-3e^{-1} + e^2 - e$

(e) $4e^{-1} - 2e^2 - 3e$

7. By using the method of variation of parameters, a particular solution y_p of the differential equation

$$y'' + y = \cos^2 x$$

is

(a) $y_p = \frac{1}{3}\cos^4 x + \sin^2 x - \frac{1}{3}\sin^4 x$ ________ (correct)

(b) $y_p = \frac{1}{3}\cos^4 x + \sin^3 x - \frac{1}{3}\sin^4 x$

(c) $y_p = \frac{1}{4}\cos^4 x + \sin^2 x - \frac{1}{4}\sin^4 x$

(d) $y_p = -\frac{1}{3}\cos^3 x + \sin^2 x + \frac{1}{3}\sin^3 x$

(e) $y_p = \frac{1}{3}\cos^4 x + \frac{1}{2}\sin^3 x - \frac{1}{3}\sin^2 x$

8. Let

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Using Cayley-Hamilton Theorem, $A^{-1} =$

(a) $\frac{1}{6}(A^2 - 6A + 11I)$ ________ (correct)

(b) $\frac{1}{11}(A^3 - 6A + 11I)$

(c) $\frac{1}{6}(A^3 - 6A^2 + 11I)$

(d) $\frac{1}{11}(A^2 - 6A + 11I)$

(e) $A^3 - 6A^2 + 11A - 6I$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & -1 \\ 0 & 3-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix}$$

$= (1-\lambda)(3-\lambda)(2-\lambda)$

$= (1-\lambda)(\lambda^2 - 5\lambda + 6)$

$= \lambda^2 - 5\lambda + 6 - \lambda^3 + 5\lambda^2 - 6\lambda$

$p(\lambda) = -\lambda^3 + 6\lambda^2 - 11\lambda + 6$

$p(A) = -A^3 + 6A^2 - 11A + 6I = 0$

$\Rightarrow 6I = A^3 - 6A^2 + 11A = A(A^2 - 6A + 11I)$

$\Rightarrow 6A^{-1} = A^2 - 6A + 11I$

$\Rightarrow A^{-1} = \frac{1}{6}(A^2 - 6A + 11I)$

9. If the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of the matrix

$$A = \begin{bmatrix} 3 & 5 & 8 \\ 0 & 7 & 0 \\ 1 & 6 & 1 \end{bmatrix}$$

are ordered as follows: $\lambda_1 < \lambda_2 < \lambda_3$, then

$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 5 & 8 \\ 0 & 7-\lambda & 0 \\ 1 & 6 & 1-\lambda \end{vmatrix} \leftarrow$

$= (7-\lambda)\left[(3-\lambda)(1-\lambda) - 8\right]$

$= (7-\lambda)(\lambda^2 - 4\lambda - 5)$

$= (7-\lambda)(\lambda - 5)(\lambda+1)$

(a) $\lambda_3 - \lambda_2 = 2$ ________ (correct)

(b) $\lambda_1 + \lambda_2 = 3$

$\Rightarrow \lambda = 7, 5, -1$

(c) $\lambda_1 + \lambda_3 = -5$

$\underset{\lambda_1}{-1} < \underset{\lambda_2}{5} < \underset{\lambda_3}{7}$

(d) $\lambda_1 + \lambda_2 + \lambda_3 = 10$

(e) $\lambda_1 \lambda_2 \lambda_3 = -30$

10. Let

$$A = \begin{bmatrix} 3 & 6 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

An eigenvector corresponding to the eigenvalue $\lambda = 3$ of A is

(a) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ ________ (correct)

(b) $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

(e) $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

11. Let $A = \begin{bmatrix} 5 & -3 \\ 2 & 0 \end{bmatrix}$. A diagonalizing matrix P, such that $P^{-1}AP$ is diagonal, is

$\det(A-\lambda I) = \begin{vmatrix} 5-\lambda & -3 \\ 2 & -\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3) = 0 \Rightarrow \lambda = 2,\ \lambda = 3$

(a) $P = \begin{bmatrix} -1 & 9 \\ -1 & 6 \end{bmatrix}$ (correct)

(b) $P = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$

(c) $P = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix}$

(d) $P = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

(e) $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$\underline{\lambda = 2}$: $\left[\begin{array}{cc|c} 3 & -3 & 0 \\ 2 & -2 & 0 \end{array}\right] \Rightarrow v_1 = v_2 = \alpha$

$v = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} \xrightarrow{\alpha = -1} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = v_1$

$\underline{\lambda = 3}$: $\left[\begin{array}{cc|c} 2 & -3 & 0 \\ 2 & -3 & 0 \end{array}\right] \Rightarrow v_1 = \frac{3}{2} v_2 = \frac{3}{2}\alpha$

$v = \begin{bmatrix} \frac{3}{2}\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} \xrightarrow{\alpha = 6} v_2 = \begin{bmatrix} 9 \\ 6 \end{bmatrix}$

$P = [v_1\ v_2] = \begin{bmatrix} -1 & 9 \\ -1 & 6 \end{bmatrix}$

12. The differential equation

$$x''' + 3x'' + 2x' - 5x = t^2$$

is equivalent to the system of first-order equations

Let $x_1 = x$, $x_2 = x'$, $x_3 = x''$

Then $x_1' = x_2$

$x_2' = x_3$

$x_3' = x''' = 5x - 2x' - 3x'' + t^2$

$= 5x_1 - 2x_2 - 3x_3 + t^2$

(a) $x_1' = x_2,\ x_2' = x_3,\ x_3' = 5x_1 - 2x_2 - 3x_3 + t^2$ (correct)

(b) $x_1' = x_3,\ x_2' = x_2,\ x_3' = 5x_1 - 2x_2 - 3x_3 + t^2$

(c) $x_1' = x_2,\ x_2' = x_3,\ x_3' = -5x_1 + 2x_2 + 3x_3 - t^2$

(d) $x_1' = x_1,\ x_2' = x_2,\ x_3' = x_3 + t^2$

(e) $x_1' = x_2,\ x_2' = x_3,\ x_3' = 5x_1 - 2x_2 - 3x_3$

13. If $X = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$ is the solution of the initial-value problem

$$X' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} X, \; X(0) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

then $c_1^2 + c_2^2 =$

$X(0) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} c_1 \\ -c_1 \end{bmatrix} + \begin{bmatrix} c_2 \\ c_2 \end{bmatrix}$

(a) 17 ____________________ (correct)

(b) 13

(c) 5

(d) 41

(e) 32

$\Rightarrow c_1 + c_2 = 3$

$-c_1 + c_2 = 5$

$2c_2 = 8 \Rightarrow \boxed{c_2 = 4}$

$\Rightarrow c_1 + 4 = 3 \Rightarrow \boxed{c_1 = -1}$

$c_1^2 + c_2^2 = 1 + 16 = 17$

14. If the general solution of

$$X' = \begin{bmatrix} 2 & 0 & 0 \\ 2 & -2 & -1 \\ -2 & 6 & 3 \end{bmatrix} X$$

is

$$X = c_1 \begin{bmatrix} 0 \\ -1 \\ a \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ b \\ 3 \end{bmatrix} e^{t} + c_3 \begin{bmatrix} c \\ 0 \\ 2 \end{bmatrix} e^{2t},$$

$\Rightarrow$ Eigenvalues : $\lambda = 0, 1, 2$

Find Corresponding eigenvectors

Then $a + b + c =$

$\Rightarrow a = 2, \; b = -1, \; c = 1$

(a) 2 ____________________ (correct)

(b) -2

(c) 1

(d) -1

(e) 0

15. A 2×2 matrix A has an eigenvector $V = \begin{bmatrix} 1 \\ i \end{bmatrix}$ associated to the eigenvalue $\lambda = 4-3i$ of A. Then the general solution of the system $X' = AX$ is

(a) $X = c_1 \begin{bmatrix} \cos(3t) \\ \sin(3t) \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} -\sin(3t) \\ \cos(3t) \end{bmatrix} e^{4t}$ ______(correct)

(b) $X = c_1 \begin{bmatrix} \cos(4t) \\ \sin(4t) \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} -\sin(4t) \\ \cos(4t) \end{bmatrix} e^{-3t}$

(c) $X = c_1 \begin{bmatrix} \cos(3t) \\ \sin(3t) \end{bmatrix} e^{-4t} + c_2 \begin{bmatrix} -\sin(3t) \\ \cos(3t) \end{bmatrix} e^{-4t}$

(d) $X = c_1 \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} 2\sin(3t) \\ -\cos(3t) \end{bmatrix} e^{4t}$

(e) $X = c_1 \begin{bmatrix} 2\cos(3t) \\ -\sin(3t) \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} -\sin(3t) \\ 2\cos(3t) \end{bmatrix} e^{4t}$

$X = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(4-3i)t}$

$= \begin{bmatrix} 1 \\ i \end{bmatrix} e^{4t} \cdot (\cos(3t) - i\sin(3t))$

$= \begin{bmatrix} \cos(3t) - i\sin(3t) \\ \sin(3t) + i\cos(3t) \end{bmatrix} e^{4t}$

$X_1 = Re(X) = \begin{bmatrix} \cos(3t) \\ \sin(3t) \end{bmatrix} e^{4t}$; $X_2 = Im(X) = \begin{bmatrix} -\sin(3t) \\ \cos(3t) \end{bmatrix} e^{4t}$

G.S.: $X = c_1 X_1 + c_2 X_2$

16. Let

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & 2 & 3 \end{bmatrix} = 3I + N, \quad N = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

$N^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & 0 & 0 \end{bmatrix}$, $N^3 = O$

If $e^{At} = e^{at} \begin{bmatrix} 1 & 0 & 0 \\ f(t) & 1 & 0 \\ g(t) & h(t) & 1 \end{bmatrix}$, then $a + f(1) + g(1) + h(1) =$

$e^{At} = e^{3It} \cdot e^{Nt} = e^{3t} I\, e^{Nt} = e^{3t} \cdot e^{Nt}$

(a) 10 ______(correct)

(b) 15

(c) 8

(d) 12

(e) 9

$= e^{3t} \begin{bmatrix} 1 & 0 & 0 \\ 2t & 1 & 0 \\ t+2t^2 & 2t & 1 \end{bmatrix}$

$a = 3$, $f(t) = 2t$, $g(t) = t + 2t^2$, $h(t) = 2t$

$a + f(1) + g(1) + h(1) = 3 + 2 + 3 + 2 = 10$

17. A possible fundamental matrix for the system

$$X' = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} X$$

is

$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 - 8$
$= \lambda^2 - 4\lambda - 5$
$= (\lambda-5)(\lambda+1)$

$\lambda = -1, \quad \lambda = 5$

(a) $\Phi(t) = \begin{bmatrix} -e^{-t} & e^{5t} \\ e^{-t} & 2e^{5t} \end{bmatrix}$ (correct)

(b) $\Phi(t) = \begin{bmatrix} e^{-t} & 0 \\ -e^{-t} & e^{5t} \end{bmatrix}$

(c) $\Phi(t) = \begin{bmatrix} 2e^{-t} & 2e^{5t} \\ e^{-t} & 4e^{5t} \end{bmatrix}$

(d) $\Phi(t) = \begin{bmatrix} -3e^{-t} & e^{5t} \\ 3e^{-t} & -e^{5t} \end{bmatrix}$

(e) $\Phi(t) = \begin{bmatrix} -e^{-t} & -e^{5t} \\ 2e^{-t} & 2e^{5t} \end{bmatrix}$

$\lambda = -1$ $\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 4 & 4 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$v_1 = -v_2$

$V = \begin{bmatrix} -\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix} \xrightarrow{\alpha=1} V_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \to X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$

$\lambda = 5$ $\left[\begin{array}{cc|c} -4 & 2 & 0 \\ 4 & -2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -4 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$v_1 = \frac{1}{2} v_2$

$V = \begin{bmatrix} \frac{1}{2}\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \xrightarrow{\alpha=2} V_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \to X_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{5t}$

$\Phi(t) = [X_1 \; X_2] = \begin{bmatrix} -e^{-t} & e^{5t} \\ e^{-t} & 2e^{5t} \end{bmatrix}$

18. Let

$$F(t) = \begin{bmatrix} \sec t \\ 0 \end{bmatrix}, \; e^{At} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}, \; 0 < t < \frac{\pi}{2}.$$

A particular solution for the system $X' = AX + F(t)$ is

$X_p = e^{At} \int e^{-At} F(t)\, dt$

(a) $X_p = \begin{bmatrix} t\cos t - \sin t \cdot \ln \cos t \\ t\sin t + \cos t \cdot \ln \cos t \end{bmatrix}$ (correct)

(b) $X_p = \begin{bmatrix} -t\cos t + \sin t \cdot \ln \cos t \\ \sin t - \cos t \cdot \ln \cos t \end{bmatrix}$

(c) $X_p = \begin{bmatrix} t\sin t - \cos t \cdot \ln \cos t \\ t\cos t + \sin t \cdot \ln \cos t \end{bmatrix}$

(d) $X_p = \begin{bmatrix} \cos t - \sin t \cdot \ln \cos t \\ \sin t + \cos t \cdot \ln \cos t \end{bmatrix}$

(e) $X_p = \begin{bmatrix} t\cos t + \sin t \cdot \ln \cos t \\ -t\sin t - \cos t \cdot \ln \cos t \end{bmatrix}$

19. The matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix}$$

$v_2 = (A-I)v_3 = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$v_1 = (A-I)v_2 = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$

has a defective eigenvalue $\lambda = 1$ of defect 2. If we choose $v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ such that $(A - I)^3 v_3 = 0$, then the general solution of $X' = AX$ is

(a) $X = \left(c_1 \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 2t \\ 1 \\ -t \end{bmatrix} + c_3 \begin{bmatrix} t^2+1 \\ t \\ -t^2/2 \end{bmatrix} \right) e^t$ ______ (correct)

(b) $X = \left(c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 2t \\ 1 \\ -t \end{bmatrix} + c_3 \begin{bmatrix} t^2 \\ t \\ -t^2 \end{bmatrix} \right) e^t$

(c) $X = \left(c_1 \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} t \\ 0 \\ -t \end{bmatrix} + c_3 \begin{bmatrix} t^2+1 \\ t \\ -t^2 \end{bmatrix} \right) e^t$

(d) $X = \left(c_1 \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 2t \\ 1 \\ -t \end{bmatrix} + c_3 \begin{bmatrix} t^2+1 \\ t \\ t^2/2 \end{bmatrix} \right) e^t$

(e) $X = \left(c_1 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2t \\ -1 \\ t \end{bmatrix} + c_3 \begin{bmatrix} t^2+1 \\ -t \\ -t^2/2 \end{bmatrix} \right) e^t$

$X_1 = v_1 e^t = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} e^t$

$X_2 = v_1 t e^t + v_2 e^t = \begin{bmatrix} 2t \\ 1 \\ -t \end{bmatrix} e^t$

$X_3 = v_1 \frac{t^2}{2} e^t + v_2 t e^t + v_3 e^t = \begin{bmatrix} t^2+1 \\ t \\ -t^2/2 \end{bmatrix} e^t$

$X = c_1 X_1 + c_2 X_2 + c_3 X_3$

20. A 2×2 matrix A has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$ with eigenvectors $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$, respectively. If

$$A^2 = \begin{bmatrix} x & y \\ z & w \end{bmatrix},$$

then $x + y + z + w =$

(a) 2 ______ (correct)

(b) -5

(c) 4

(d) -3

(e) 9

$P = \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$P^{-1} = \frac{1}{-1}\begin{bmatrix} 3 & -4 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 1 & -1 \end{bmatrix}$

"diagonalization"

$A = PDP^{-1}$

$= \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -3 & 4 \\ 1 & -1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 8 \\ 1 & 6 \end{bmatrix}\begin{bmatrix} -3 & 4 \\ +1 & -1 \end{bmatrix}$

$= \begin{bmatrix} 5 & -4 \\ 3 & -2 \end{bmatrix}$

$A^2 = \begin{bmatrix} 5 & -4 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 5 & -4 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 13 & -12 \\ 9 & -8 \end{bmatrix}$

$x+y+z+w = 13 - 12 + 9 - 8 = 1 + 1 = 2$

21. By the method of undetermined coefficients, a particular solution for the DE $y'' + 2y' + 2y = 2\sin x$ has the form $y_p = A\sin x + B\cos x$.
(Answer True or False by filling in the OMR sheet)

(a) True ——————————————————(correct)

(b) False

22. The vectors $u = (1,2,3)$, $v = (4,5,6)$, $w = (7,8,9)$ of $\mathbb{R}^3$ are linearly independent.
(Answer True or False by filling in the OMR sheet)

(a) False ——————————————————(correct)

(b) True

$$\begin{array}{ccc} u & v & w \\ \end{array}$$
$$\rightarrow \begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix}$$

$= 1(45-48) - 4(18-24) + 7(12-15)$

$= -3 - 4(-6) + 7(-3)$

$= -3 + 24 - 21$

$= 0 \implies$ L. Dep.

23. The following matrix is nilpotent:

$$B = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$B^2 = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B^3 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

(Answer True or False by filling in the OMR sheet)

(a) True ________________ (correct)

(b) False

24. If λ is a repeated eigenvalue of some matrix, then it is a defective eigenvalue. (Answer True or False by filling in the OMR sheet)

(a) False ________________ (correct)

(b) True

If λ is repeated twice (say), then it may have $\underline{2}$ L. indep. eigenvectors.

25. The vector $v = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ is an eigenvector of the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 4 & 3 & 5 \\ -1 & 0 & -1 \end{bmatrix}.$$

(Answer True or False by filling in the OMR sheet)

(a) False ______________________________ (correct)

(b) True

Check the defn: $Av = \lambda v$ for some λ

$$Av = \begin{bmatrix} 1 & 1 & 2 \\ 4 & 3 & 5 \\ -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix} \neq \lambda \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \text{ for any } \lambda$$

So v is not an eigenvector of A.