1. If $y = x^m$ is a solution of the differential equation

$$x^2y'' + 5xy' + 4y = 0,$$

then $m =$

See Q11 Page 8

(a) -2 ____(correct)

(b) -1

(c) 1

(d) 3

(e) -3

$y = x^m \Rightarrow y' = m x^{m-1} \Rightarrow y'' = m(m-1)x^{m-2}$

$x^2y'' + 5xy' + 4y = 0 \Rightarrow$

$m(m-1)x^m + 5m x^m + 4x^m = 0$

$\Rightarrow x^m\left[m^2 + 4m + 4\right] = 0$

$\Rightarrow (m+2)^2 = 0 \Rightarrow \boxed{m = -2}$

2. A particle is moving along a straight line with position function $x(t)$. If the acceleration of the particle is $a(t) = 50\sin(5t)$, the initial position is $x(0) = 8$, and the initial velocity is $v(0) = -10$, then $x\left(\frac{\pi}{2}\right) =$

See Q18 Page 17

(a) 6 ____(correct)

(b) 10

(c) 8

(d) 12

(e) 4

$a(t) = 50\sin(5t) \Rightarrow v(t) = \int 50\sin(5t)\,dt$

$\Rightarrow v(t) = -10\cos(5t) + C_1$

$v(0) = -10 \Rightarrow -10 + C_1 = -10 \Rightarrow C_1 = 0$

$\therefore v(t) = -10\cos(5t)$

$\Rightarrow s(t) = \int -10\cos(5t)\,dt$

$\Rightarrow s(t) = -2\sin(5t) + C_2, \quad x(0) = 8 \Rightarrow C_2 = 8$

$\therefore s(t) = -2\sin(5t) + 8 \Rightarrow s\left(\frac{\pi}{2}\right) = 6$

3. Which one of the following differential equation is an exact equation

See Q31-42 Page 74

(a) $\left(y^3 + \frac{x}{y}\right) dx + \left(3xy^2 - \frac{x^2}{2y^2}\right) dy = 0$ ____________________(correct)

(b) $(4x + y)\,dx + (6y - x)\,dy = 0$

(c) $(x^2 + 2xy^2)\,dx + (4xy + 2xy^2)\,dy = 0$

(d) $(\cos x + \ln y)\,dx + \left(\frac{x}{y} + \sin x\right) dy = 0$

(e) $(xy + e^y)\,dx + (xy + e^x)\,dy = 0$

Answer is

$$\left(y^3 + \frac{x}{y}\right) dx + \left(3xy^2 - \frac{x^2}{2y^2}\right) dy = 0$$

$$M(x,y) = y^3 + \frac{x}{y} \Rightarrow M_y = 3y^2 - \frac{x}{y^2}$$

$$N(x,y) = 3xy^2 - \frac{x^2}{2y^2} \Rightarrow N_x = 3y^2 - \frac{x}{y^2}$$

Since $M_y = N_x$ it is an Exact equation.

4. If $y = f(x)$ is the solution of the initial value problem

$$\frac{dy}{dx} = \frac{20}{x^2 + 16}, \quad y(0) = 0$$

then $y(4) =$

See Q7 page 17

(a) $\frac{5\pi}{4}$ ____________________(correct)

(b) $\frac{3\pi}{4}$

(c) $\frac{7\pi}{4}$

(d) 0

(e) $\frac{\pi}{4}$

$$y(x) = \int \frac{20}{x^2+16}\,dx$$

$$= 20\left(\frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right)\right) + C$$

$$= 5\tan^{-1}\left(\frac{x}{4}\right) + C$$

$$y(0) = 0 \Rightarrow C = 0$$

$$\therefore\ y(x) = 5\tan^{-1}\frac{x}{4} \Rightarrow y(4) = \frac{5\pi}{4}$$

5. If $y(x)$ is the solution of the initial value problem

$$x\frac{dy}{dx} - y = 2x^2y,\ y(1) = 1$$

then, $y(2) =$

see Q35 page 43

The DE is separable

(a) $2e^3$ (correct)

(b) $2e^4$

(c) e^4

(d) 0

(e) $5e^3$

$$x\frac{dy}{dx} = y + 2x^2y \Leftrightarrow x\frac{dy}{dx} = y(1+2x^2)$$

$$\Rightarrow \frac{dy}{y} = \frac{1+2x^2}{x}dx$$

$$\Rightarrow \frac{dy}{y} = \left(\frac{1}{x} + 2x\right)dx \Rightarrow \ln|y| = \ln|x| + x^2 + c_1$$

$$\Rightarrow \ln\left|\frac{y}{x}\right| = x^2 + c_1 \Rightarrow \frac{y}{x} = c\,e^{x^2} \Rightarrow y = c\,x\,e^{x^2}$$

$$y(1) = 1 \Rightarrow c\,e = 1 \Rightarrow c = e^{-1} \Rightarrow y = x\,e^{x^2-1} \Rightarrow y(2) = 2e^3$$

6. The general solution of the linear differential equation

$$\frac{dy}{dx} + 3y = 2xe^{-3x}$$

is given by

See Q3 page 56

(a) $y(x) = e^{-3x}(x^2 + c)$ (correct)

(b) $y(x) = e^{3x}(x^2 + c)$

(c) $y(x) = e^{3x}(x^3 + c)$

(d) $y(x) = e^{-3x}(x^3 + c)$

(e) $y(x) = e^{3x}(2x^2 + c)$

Integrating factor : $u(x) = e^{\int 3dx} = e^{3x}$

$$\frac{d}{dx}\left(e^{3x}y\right) = 2x$$

$$\Rightarrow e^{3x}y = x^2 + c$$

$$\Rightarrow y = e^{-3x}(x^2 + c)$$

7. The matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$ is invertible if

See Example 8 page 210

(a) $c \neq 0,\ b \neq 0,\ c \neq b$ ______ (correct)
(b) $c \neq b,\ b \neq 0$
(c) $c \neq 0,\ b \neq 0,\ c = b$
(d) $c = 0,\ b \neq 0,\ c = b$
(e) $c \neq 0,\ b = 0,\ c \neq b$

A is invertible $\Leftrightarrow |A| \neq 0$

$$\Rightarrow \begin{vmatrix} 1 & 0 & 0 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \neq 0$$

$$\Rightarrow \begin{vmatrix} b & b^2 \\ c & c^2 \end{vmatrix} \neq 0 \Rightarrow bc^2 - cb^2 \neq 0$$

$$\Rightarrow cb(c-b) \neq 0 \Rightarrow c \neq 0,\ b \neq 0,\ c \neq b.$$

8. A certain city had a population of 25000 in 1960 and a popultaion of 30,000 in 1970. Assume that its population will continue to grow exponentially at a constant rate. The population in 1980 is

See Q33 page 44

(a) 36000 ______ (correct)
(b) 38000
(c) 34000
(d) 42000
(e) 48000

Let the population at any time be $P(t)$.

$P(t) = C e^{Kt}$. $P(0) = 25000$, $P(10) = 30000$

$P(0) = 25000 \Rightarrow C = 25000 \Rightarrow P(t) = 25000\, e^{Kt}$

$P(10) = 30000 \Rightarrow 30000 = 25000\, e^{10K} \Rightarrow e^{10K} = \frac{6}{5}$

$10K = \ln\left(\frac{6}{5}\right) \Rightarrow K = \frac{1}{10}\ln\left(\frac{6}{5}\right) \Rightarrow P(t) = 25000\, e^{\frac{t}{10}\ln\left(\frac{6}{5}\right)}$

$P(20) = 25000\, e^{2\ln\left(\frac{6}{5}\right)} = 25000\left(\frac{36}{25}\right) = 36000$

9. (11 points)Given the system $\begin{cases} 3x - y - 5z = 3 \\ 4x - 4y - 3z = -4 \\ x - 5z = 2 \end{cases}$

See $Q_{11}/165$

Solve the system by using Gaussian elimination method.
(No other method is accepted)

$$\left[\begin{array}{ccc|c} 3 & -1 & -5 & 3 \\ 4 & -4 & -3 & -4 \\ 1 & 0 & -5 & 2 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 0 & -5 & 2 \\ 4 & -4 & -3 & -4 \\ 3 & -1 & -5 & 3 \end{array}\right]$$

(2pts)

$$\xrightarrow[-4R_1+R_2]{-3R_1+R_3} \left[\begin{array}{ccc|c} 1 & 0 & -5 & 2 \\ 0 & -4 & 17 & -12 \\ 0 & -1 & 10 & -3 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 0 & -5 & 2 \\ 0 & -1 & 10 & -3 \\ 0 & -4 & 17 & -12 \end{array}\right]$$

(3pts)

$$\xrightarrow{-R_2} \left[\begin{array}{ccc|c} 1 & 0 & -5 & 2 \\ 0 & 1 & -10 & 3 \\ 0 & -4 & 17 & -12 \end{array}\right] \xrightarrow{4R_2+R_3} \left[\begin{array}{ccc|c} 1 & 0 & -5 & 2 \\ 0 & 1 & -10 & 3 \\ 0 & 0 & -23 & 0 \end{array}\right]$$

(2 pts)

$$\xrightarrow{\frac{1}{-23}R_3} \left[\begin{array}{ccc|c} 1 & 0 & -5 & 2 \\ 0 & 1 & -10 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\Rightarrow x - 5z = 2$

$y - 10z = 3$

$z = 0 \Rightarrow y = 3, \ x = 2$ (4 pts)

The solution set $= \{(2, 3, 0)\}$

See Q_{33} | Page 216

10. (11 points) Let $A = \begin{bmatrix} -5 & -2 & 2 \\ 1 & 5 & -3 \\ 5 & -3 & 1 \end{bmatrix}$. Find the inverse of A only by using the adjoint formula.

$$|A| = \begin{vmatrix} -5 & -2 & 2 \\ 1 & 5 & -3 \\ 5 & -3 & 1 \end{vmatrix} = -5\begin{vmatrix} 5 & -3 \\ -3 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & -3 \\ 5 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & 5 \\ 5 & -3 \end{vmatrix}$$

$$\Rightarrow |A| = -5(-4) + 2(16) + 2(-28) = -4 \quad (2\text{pts})$$

$$A_{11} = \begin{vmatrix} 5 & -3 \\ -3 & 1 \end{vmatrix} = -4, \quad A_{12} = -\begin{vmatrix} 1 & -3 \\ 5 & 1 \end{vmatrix} = -16, \quad A_{13} = \begin{vmatrix} 1 & 5 \\ 5 & -3 \end{vmatrix} = -28$$

$$A_{21} = -\begin{vmatrix} -2 & 2 \\ -3 & 1 \end{vmatrix} = -4, \quad A_{22} = \begin{vmatrix} -5 & +2 \\ 5 & 1 \end{vmatrix} = -15, \quad A_{23} = -\begin{vmatrix} -5 & -2 \\ 5 & -3 \end{vmatrix} = -25$$

$$A_{31} = \begin{vmatrix} -2 & 2 \\ 5 & -3 \end{vmatrix} = -4, \quad A_{32} = -\begin{vmatrix} -5 & 2 \\ 1 & -3 \end{vmatrix} = -13, \quad A_{33} = \begin{vmatrix} -5 & -2 \\ 1 & 5 \end{vmatrix} = -23$$

$$[A_{ij}] = \begin{bmatrix} -4 & -16 & -28 \\ -4 & -15 & -25 \\ -4 & -13 & -23 \end{bmatrix} \quad (6\text{pts})$$

$$A^{-1} = \frac{1}{|A|}[A_{ij}]^T = \frac{1}{-4}\begin{bmatrix} -4 & -4 & -4 \\ -16 & -15 & -13 \\ -28 & -25 & -23 \end{bmatrix}$$

(2pts)

$$= \begin{bmatrix} 1 & 1 & 1 \\ 4 & \frac{15}{4} & \frac{13}{4} \\ 7 & \frac{25}{4} & +\frac{23}{4} \end{bmatrix} \quad (1\text{pt})$$

11. (11 points) Verify that the differential equation See Q31-42 Page 74

$$[\sin(x^2+y)+2x^2\cos(x^2+y)]\,dx+[x\cos(x^2+y)+2ye^{y^2}]\,dy=0$$

is exact and then solve it.

$M(x,y) = \sin(x^2+y) + 2x^2\cos(x^2+y)$

$M_y = \cos(x^2+y) - 2x^2\sin(x^2+y)$ (1 pt)

(1 pt)

$N(x,y) = x\cos(x^2+y) + 2ye^{y^2} \Rightarrow N_x = \cos(x^2+y) - 2x^2\sin(x^2+y)$

Since $M_y = N_x \Rightarrow$ the DE is Exact. (1 pt)

$\therefore$ There is a function $F(x,y)$ s.t $\frac{\partial F}{\partial x} = M$, $\frac{\partial F}{\partial y} = N$.

$\frac{\partial F}{\partial y} = x\cos(x^2+y) + 2ye^{y^2}$

$\Rightarrow F(x,y) = \int [x\cos(x^2+y) + 2ye^{y^2}]\,dy$

$= x\sin(x^2+y) + e^{y^2} + g(x)$ (3 pts)

$\frac{\partial F}{\partial x} = M \Rightarrow \cancel{\sin(x^2+y)} + \cancel{2x^2\cos(x^2+y)} + g'(x)$

$= \cancel{\sin(x^2+y)} + \cancel{2x^2\cos(x^2+y)}$ (3 pts)

$\therefore$ $g'(x) = 0 \Rightarrow g(x) = C_1$.

$\therefore$ $F(x,y) = x\sin(x^2+y) + e^{y^2} + C_1$ (1 pt)

The solution of the DE is

$x\sin(x^2+y) + e^{y^2} = C$. (1 pt)

12. (11 points) Solve the initial value problem

see Q_{25} page 56

$$(x^2+1)\frac{dy}{dx} + 3x^3y = 6xe^{-\frac{3}{2}x^2},\ y(0) = 1.$$

$$\frac{dy}{dx} + \frac{3x^3}{x^2+1}\,y = \frac{6x}{x^2+1}\,e^{-\frac{3}{2}x^2}$$

Integrating factor $= u(x) = e^{\int \frac{3x^3}{x^2+1}dx}$ (2pts)

$$\Rightarrow u(x) = e^{\int\left(3x - \frac{3x}{x^2+1}\right)dx} = e^{\frac{3}{2}x^2}\cdot e^{-\frac{3}{2}\ln(x^2+1)}$$

$$= (x^2+1)^{-\frac{3}{2}}\cdot e^{\frac{3}{2}x^2} \quad \text{(3pts)}$$

$$\Rightarrow \frac{d}{dx}\left((x^2+1)^{-\frac{3}{2}} e^{\frac{3}{2}x^2} y\right) = 6x(x^2+1)^{-\frac{5}{2}} \quad \text{(2pts)}$$

$$\Rightarrow (x^2+1)^{-\frac{3}{2}} e^{\frac{3}{2}x^2} y = \cancel{3}(x^2+1)^{-\frac{3}{2}}\cdot\frac{(-2)}{\cancel{3}} + C \quad \text{(2pts)}$$

$$\Rightarrow y = -2e^{-\frac{3}{2}x^2} + C\,e^{-\frac{3}{2}x^2}(x^2+1)^{3/2}$$

$y(0) = 1 \Rightarrow C = 3$ (1pt)

$\Rightarrow$ The solution of the IVP is

$$y = e^{-\frac{3}{2}x^2}\left[-2 + 3\sqrt{(x^2+1)^3}\right].\quad \text{(1 pt)}$$

Answer KEY

Q	MASTER	CODE01	CODE02	CODE03	CODE04
1	A	E 6	B 2	D 2	E 4
2	A	D 3	D 7	A 1	A 7
3	A	C 4	C 1	A 5	C 3
4	A	C 5	C 5	E 7	B 2
5	A	A 8	C 6	A 6	D 8
6	A	C 7	A 4	A 4	B 1
7	A	C 2	E 3	A 3	E 5
8	A	A 1	B 8	A 8	B 6
9		11	9	9	9
10		9	11	10	10
11		12	12	11	11
12		10	10	12	12