1. Given that vectors $\vec{t} = (2, -7, 9)$, $\vec{u} = (1, -2, 2)$, $\vec{v} = (3, 0, 1)$ and $\vec{w} = (1, -1, 2)$. If $\vec{t} = a\vec{u} + b\vec{v} + c\vec{w}$, then $a^2 + b^2 + c^2 =$

Q25/4.1

(a) 14 ______________________________ (correct)
(b) 12
(c) 16
(d) 13
(e) 10

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ -2 & 0 & -1 & -7 \\ 2 & 1 & 2 & 9 \end{array}\right] \xrightarrow[-2R_1+R_3]{2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 0 & 6 & 1 & -3 \\ 0 & -5 & 0 & 5 \end{array}\right]$$

$$\begin{cases} a + 3b + c = 2 \\ 6b + c = -3 \\ -5b = 5 \end{cases} \Rightarrow b = -1, \quad c = -3 - 6b = 3$$

and $a = -3b - c + 2 = 2$

$$\therefore \quad a^2 + b^2 + c^2 = 4 + 1 + 9 = 14$$

2. Let $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_3 = 0\}$. Which one of the following is true

Q1/4.2

(a) S is a subspace of $\mathbb{R}^3$ ______________________________ (correct)
(b) S is not closed under vector additions but closed under multiplication by scalar
(c) S is closed under vector addition but not closed under multiplication by scalar
(d) S is neither closed under vector addition or closed under multiplication by scalar
(e) The vector $(0, 1, 2) \in S$

let $v_1 = (x_1, x_2, 0) \in S$

$v_2 = (y_1, y_2, 0) \in S$

$$\left.\begin{array}{l} v_1 + v_2 = (x_1 + y_1, x_2 + y_2, 0) \in S \\ c v_1 = (c x_1, c x_2, 0) \in S \end{array}\right\} \therefore S \text{ is a subspace of } \mathbb{R}^3$$

3. The values of k for which the three vectors

$$v_1 = (1,0,1),\ v_2 = (2,-3,4),\ \text{and}\ v_3 = (3,5,k)$$

form a basis for $\mathbb{R}^3$ are

Q17 / 4.3

(a) $k \neq -\frac{1}{3}$ ______________________________ (correct)

(b) $k \neq 2$

(c) $k \neq 0$

(d) $k \neq 1$

(e) $k \neq 5$

The three vectors will form a basis of $\mathbb{R}^3$ if they are Linearly Independent $\Leftrightarrow$

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & 5 \\ 1 & 4 & k \end{vmatrix} \neq 0 \Leftrightarrow -3k - 20 + 10 + 9 \neq 0$$

$$\Leftrightarrow -3k \neq 1 \Leftrightarrow k \neq -\frac{1}{3}$$

4. The rank of the matrix $A = \begin{bmatrix} 1 & 3 & 3 & 9 \\ 2 & 7 & 4 & 8 \\ 2 & 7 & 5 & 12 \\ 2 & 8 & 3 & 2 \end{bmatrix}$ is

Q9 / 4.5

(a) 3 ______________________________ (correct)

(b) 4

(c) 2

(d) 1

(e) 0

$$\begin{bmatrix} 1 & 3 & 3 & 9 \\ 2 & 7 & 4 & 8 \\ 2 & 7 & 5 & 12 \\ 2 & 8 & 3 & 2 \end{bmatrix} \xrightarrow[\substack{-2R_1+R_3 \\ -2R_1+R_4}]{-2R_1+R_2} \begin{bmatrix} 1 & 3 & 3 & 9 \\ 0 & 1 & -2 & -10 \\ 0 & 1 & -1 & -6 \\ 0 & 2 & -3 & -16 \end{bmatrix}$$

$$\xrightarrow[-2R_2+R_4]{-R_2+R_3} \begin{bmatrix} 1 & 3 & 3 & 9 \\ 0 & 1 & -2 & -10 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{-R_3+R_4} \begin{bmatrix} 1 & 3 & 3 & 9 \\ 0 & 1 & -2 & -10 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ The Rank$(A) = 3$

5. The Wronskian of the functions

$$f(x) = 1,\ g(x) = x,\ h(x) = x^2$$

is

Q7 / 5.2

(a) 2 ________________________________ (correct)
(b) $2x$
(c) x^2
(d) 4
(e) 0

$$W(1, x, x^2) = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2$$

6. Let $y(x)$ be the solution of the initial value problem

$$y'' - 4y' + 3y = 0;\ y(0) = 7,\ y'(0) = 11$$

$y(\ln 2) =$

Q21 / 5.3

(a) 26 ________________________________ (correct)
(b) 24
(c) 22
(d) 20
(e) 18

$m^2 - 4m + 3 = 0$

$(m-1)(m-3) = 0 \Rightarrow m = 1,\ m = 3$

$\therefore\ y = c_1 e^{x} + c_2 e^{3x}$

$y' = c_1 e^{x} + 3c_2 e^{3x}$

$y(0) = 7 \Rightarrow c_1 + c_2 = 7$ $\Rightarrow 2c_2 = 4 \Rightarrow c_2 = 2$

$y'(0) = 11 \Rightarrow c_1 + 3c_2 = 11$ $\Rightarrow c_1 = 5$

$\therefore\ y(x) = 5e^{x} + 2e^{3x} \Rightarrow y(\ln 2) = 10 + 16 = 26$

7. A linear homogeneous constant-coefficient equation whose general solution is

$$y(x) = Ae^{2x} + B\cos(2x) + C\sin(2x)$$

is

Q40 / 5.3

(a) $y''' - 2y'' + 4y' - 8y = 0$ ______ (correct)
(b) $y''' + 3y'' - 4y' + 5y = 0$
(c) $y''' - 3y'' + 4y' - 5y = 0$
(d) $y''' - 4y'' + 5y' - 8y = 0$
(e) $y''' - 5y'' + 4y' + 8y = 0$

The roots of the characteristic equation are $m = 2, \pm 2i$

$(m-2)(m^2+4) = 0$

$\Rightarrow m^3 + 4m - 2m^2 - 8 = 0$

$\Rightarrow m^3 - 2m^2 + 4m - 8 = 0$

So, a DE is $y''' - 2y'' + 4y' - 8y = 0$

8. An appropriate form of a particular solution y_p for the non-homogeneous differential equation

$$y''' - y'' - 12y' = x - 2xe^{-3x}$$

is given by

Q24 / 5.5

(a) $y_p(x) = Ax + Bx^2 + (Cx + Dx^2)e^{-3x}$ ______ (correct)
(b) $y_p(x) = A + Bx + (Cx + Dx^2)e^{-3x}$
(c) $y_p(x) = Ax + Bx^2 + (C + Dx^2)e^{-3x}$
(d) $y_p(x) = A + Bx + (C + Dx^2)e^{-3x}$
(e) $y_p(x) = A + Bx^2 + (Cx + D)e^{-3x}$

Consider $y''' - y'' - 12y' = 0 \Rightarrow m^3 - m^2 - 12m = 0$

$\Rightarrow m(m^2 - m - 12) = 0 \Rightarrow m(m-4)(m+3) = 0$

$\Rightarrow m = 0, 4, -3 \Rightarrow y_c = c_1 + c_2 e^{4x} + c_3 e^{-3x}$

Try: $y_p = (A + Bx) + (C + Dx)e^{-3x}$. To remove the duplication we get
$y_p = (Ax + Bx^2) + (Cx + Dx^2)e^{-3x}$

9. **(11-points)** Find a basis for the solution space of the given homogeneous system

$$\begin{cases} x_1 - 4x_2 - 3x_3 - 7x_4 = 0 \\ 2x_1 - x_2 + x_3 + 7x_4 = 0 \\ x_1 + 2x_2 + 3x_3 + 11x_4 = 0 \end{cases}$$

Q21/4.4

(1pt)

$$\left[\begin{array}{cccc|c} 1 & -4 & -3 & -7 & 0 \\ 2 & -1 & 1 & 7 & 0 \\ 1 & 2 & 3 & 11 & 0 \end{array}\right] \xrightarrow[-R_1+R_3]{-2R_1+R_2} \left[\begin{array}{cccc|c} 1 & -4 & -3 & -7 & 0 \\ 0 & 7 & 7 & 21 & 0 \\ 0 & 6 & 6 & 18 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{7}R_2} \left[\begin{array}{cccc|c} 1 & -4 & -3 & -7 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 6 & 6 & 18 & 0 \end{array}\right] \xrightarrow{-6R_2+R_3} \left[\begin{array}{cccc|c} 1 & -4 & -3 & -7 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(1pt) (1pt) (3pts)

let $x_4 = t$, $x_3 = s$.

$x_2 = -x_3 - 3x_4 = -s - 3t$ (1pt)

$x_1 = 4x_2 + 3x_3 + 7x_4$

$= -4s - 12t + 3s + 7t = -s - 5t$ (1pt)

So, a solution of the system has the form

$\langle -s-5t, -s-3t, s, t \rangle$ (1pt)

$= s\langle -1, -1, 1, 0 \rangle + t\langle -5, -3, 0, 1 \rangle$ (1pt)

A basis for the solution space is

$\{ \langle -1, -1, 1, 0 \rangle, \langle -5, -3, 0, 1 \rangle \}$ (1pt)

10. **(11-points)** Find the general solution of the differential equation

$$(D-1)(D-2)^2(D^2+1)(D^2+6D+13)^2y=0$$

(Example 5 + 6
5.3)

The auxiliary equation is

$(m-1)(m-2)^2(m^2+1)(m^2+6m+13)^2=0$ (1 pt)

The roots are

$m = 1,\ 2$ (order 2), $\pm i$, $-3 \pm 2i$ (order 2)
(1pt) (1pt) (1pt) (1pt)

The general solution is

(1pt) (1pt) (1pt) (1pt)

$y = c_1 e^{x} + (c_2 + c_3 x)e^{2x} + c_4 \cos x + c_5 \sin x$

$+ (c_6 + c_7 x)e^{-3x}\cos(2x) + (c_8 + c_9 x)e^{-3x}\sin(2x)$

(1pt) (1pt)

11. **(11-points)** Use the method of variation of parameters to find a particular solution of the differential equation

$$y'' + 9y = 2\sec(3x).$$

Q53 / 5.5

$y'' + 9y = 0$

$m^2 + 9 = 0 \Rightarrow m = \pm 3i$

$\therefore\ y_c = c_1 \cos(3x) + c_2 \sin(3x)$ (1pt)

let $y_p = u_1(x)\cos(3x) + u_2(x)\sin(3x)$ (1pt)

$$W = \begin{vmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{vmatrix} = 3\cos^2(3x) + 3\sin^2(3x)$$

$= 3$ (2pts)

$$W_1 = \begin{vmatrix} 0 & \sin(3x) \\ 2\sec(3x) & 3\cos(3x) \end{vmatrix} = -2\tan(3x)$$ (2pts)

$$W_2 = \begin{vmatrix} \cos(3x) & 0 \\ -3\sin(3x) & 2\sec(3x) \end{vmatrix} = 2$$ (2pts)

$u_1(x) = \int \frac{W_1}{W}\,dx = \int \frac{-2}{3}\tan(3x)\,dx = \frac{-2}{9}\ln|\sec(3x)|$ (1pt)

$u_2(x) = \int \frac{2}{3}\,dx = \frac{2}{3}x$ (1pt)

$y_p = -\frac{2}{9}\cos(3x)\ln|\sec(3x)| + \frac{2}{3}x\sin(3x)$

(1pt)

12. **(11-points)** Use the method of undetermined coefficients to find the general solution of the differential equation

$$4y'' + 4y' + y = 3xe^x$$

Q4/5.5

(No other method is accepted).

$4y'' + 4y' + y = 0$

$4m^2 + 4m + 1 = 0 \Rightarrow (2m+1)^2 = 0 \Rightarrow m = -\frac{1}{2}$ (order 2)

$y_c = c_1 e^{-\frac{1}{2}x} + c_2 x e^{-\frac{1}{2}x}$ (2pts)

let $y_p = Ae^x + Bxe^x = (A+Bx)e^x$ (2pts)

$y_p' = Be^x + (A+Bx)e^x = (B+A+Bx)e^x$ (1pt)

$y_p'' = Be^x + (B+A+Bx)e^x = (2B+A+Bx)e^x$ (1pt)

Substituting in the DE,

$(8B + 4A + 4Bx)e^x + (4B + 4A + 4Bx)e^x + (A + Bx)e^x = 3xe^x$

$(12B + 9A + 9Bx)e^x = 3xe^x$ (1pt)

$\Rightarrow 12B + 9A = 0$ (1pt)

$9B = 3 \Rightarrow B = \frac{1}{3}$ (1pt) $\Rightarrow A = -\frac{4}{9}$

$\therefore y_p = \left(\frac{1}{3}x - \frac{4}{9}\right)e^x$ (1pt)

The general solution is (1pt)

$y = y_c + y_p = c_1 e^{-\frac{x}{2}} + c_2 x e^{-\frac{x}{2}} + \left(\frac{1}{3}x - \frac{4}{9}\right)e^x.$

Answer KEY

Q	MASTER	CODE01	CODE02	CODE03	CODE04
1	A	C 5	A 5	D 3	B 4
2	A	B 3	D 1	D 5	E 7
3	A	B 1	A 4	B 6	D 8
4	A	B 4	D 3	E 4	E 1
5	A	E 8	E 2	A 8	E 5
6	A	E 7	D 6	E 1	B 3
7	A	A 2	D 7	E 2	B 6
8	A	A 6	C 8	A 7	C 2
9		10	10	10	9
10		12	12	12	11
11		11	11	11	12
12		9	9	9	10