King Fahd University of Petroleum and Minerals
Department of Mathematics

Math 208

Final Exam

222

May 18, 2023

Net Time Allowed: 180 Minutes

USE THIS AS A TEMPLATE

Write your questions, once you are satisfied upload this file.

1. If $y = y(x)$ is the solution of the initial-value problem

$$xy' = 3y + x^4 \cos x,\ y(2\pi) = 0, \text{ then } \left(\frac{2}{\pi}\right)^3 y\left(\frac{\pi}{2}\right) =$$

(a) 1
(b) -1
(c) 2
(d) -2
(e) 3

2. Let $F(x, y) = e$ be the solution of the initial-value problem

$$\begin{aligned}(\cos x + \ln y)\,dx + \left(\frac{x}{y} + e^y\right)\,dy &= 0\\ y(0) &= 1\end{aligned}.$$

Then $F(0, 2)$ equals to

(a) e^2
(b) e
(c) e^3
(d) e^4
(e) e^{-1}

3. By using Cramer's Rule to solve the system

$$\begin{aligned} 5x_1 + 4x_2 - 2x_3 &= 4 \\ 2x_1 + \quad\quad + 3x_3 &= 2 \\ 2x_1 - x_2 + x_3 &= 1 \end{aligned}$$

then $x_3 =$

(a) $\frac{2}{7}$
(b) $\frac{13}{7}$
(c) $\frac{8}{7}$
(d) $\frac{5}{7}$
(e) $\frac{1}{7}$

4. Which of the following sets $s = \{(x_1, x_2, x_3)\}$ forms a subspace of $\mathbb{R}^3$?

(a) All vectors such that $(x_1)^2 + (x_2)^2 = 0$
(b) All vectors such that $x_2 = 1$
(c) All vectors such that $|x_1| + |x_3| = 1$
(d) All vectors such that $x_1 + x_2 + x_3 = 1$
(e) All vectors such that $x_1 x_2 = x_3$

5. Let the solution space of the system

$$\begin{aligned} x_1 - 4x_2 - 3x_3 - 7x_4 = 0 \\ 2x_1 - x_2 + x_3 + 7x_4 = 0 \\ x_1 + 2x_2 + 3x_3 + 11x_4 = 0 \end{aligned}$$

have the following vectors $u = (-1, -1, 1, 0)$ and $v = (a, b, 0, 1)$ as basis for the solution space. Then $b - a =$

(a) 2
(b) -2
(c) 3
(d) 1
(e) -1

6. The general solution of the differential equation

$y''' + 3y'' - 4y = 0$ is $y =$

(a) $c_1e^x + c_2e^{-2x} + c_3xe^{-2x}$
(b) $c_1e^{-x} + c_2e^{2x} + c_3xe^{2x}$
(c) $c_1e^x + c_2e^{-2x}$
(d) $c_1e^x + c_2e^{-x} + c_3xe^{-x}$
(e) $c_1e^{-x} + c_2e^{-2x} + c_3xe^{-2x}$

7. If y_p is a particular solution of the differential equation $y''+4y = 3x^3$, with $y_p(0) = 0$ then $y_p(-1) =$

(a) $\frac{3}{8}$

(b) $\frac{1}{2}$

(c) $\frac{-3}{8}$

(d) $\frac{-1}{2}$

(e) $\frac{9}{8}$

8. If $y_p = \cos x \ln|\csc x + \cot x| + u_2 \sin x$, is a particular solution of the differential equation $y'' + y = \csc^2 x$, then $u_2(x) =$

(a) $-\csc x$

(b) $-\cot x$

(c) $-\tan x$

(d) $-\sec x$

(e) $-\cos x$

9. A basis for $\mathbb{R}^2$ which consists of eigenvectors of the matrix $A = \begin{bmatrix} 10 & -8 \\ 6 & -4 \end{bmatrix}$ is

(a) $\left\{ \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$

(e) $\left\{ \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$

10. Suppose that a 2×2 matrix A has the eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 1$ and their corresponding eigenvectors $v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, respectively.
If $A^2 = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$, then $x + y + z + w =$

(a) -7

(b) -6

(c) -8

(d) -5

(e) -9

11. Given that $A = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$. If $A^{-1} = aA^2 + bA + cI$, then $12(a+b+c) =$

(a) 10
(b) 9
(c) 11
(d) 8
(e) 12

12. The differential equation $x^{(4)} + 6x'' - 3x' + x = \cos 3t$ can be transferred into a system of linear first-order differential equations of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cos 3t \end{bmatrix}$$

where $A =$

(a) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 3 & -6 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 3 & 6 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -3 & -6 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -3 & -6 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 3 & 6 & 0 \end{bmatrix}$

13. If $X = c_1 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} e^{5t}$ is the solution of the initial value problem $X' = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix} X$, $X(0) = \begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix}$, then $c_1^2 + c_2^2 =$

(a) 13
(b) 15
(c) 14
(d) 12
(e) 11

14. The solution of $X' = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix} X$, $X(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ at $t = \pi$ is equal to

(a) $\begin{pmatrix} -1 \\ -2 \end{pmatrix} e^{4\pi}$
(b) $\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4\pi}$
(c) $\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4\pi}$
(d) $\begin{pmatrix} 2 \\ 2 \end{pmatrix} e^{4\pi}$
(e) $\begin{pmatrix} -2 \\ 2 \end{pmatrix} e^{4\pi}$

15. If the general solution of $X' = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} X$ is $X = c_1 \begin{bmatrix} 1 \\ a \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} b \\ 1 \end{bmatrix} e^{\lambda t}$, then $a + b + \lambda =$

(a) 4

(b) 5

(c) 3

(d) 6

(e) 2

16. The general solution of $X' = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} X$ is $X =$

(a) $c_1 \begin{bmatrix} -3 \\ 3 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} -3t+1 \\ 3t \end{bmatrix} e^{4t}$

(b) $c_1 \begin{bmatrix} -3 \\ 3 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} 3t+2 \\ 3t \end{bmatrix} e^{4t}$

(c) $c_1 \begin{bmatrix} -3 \\ 3 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} -2t+1 \\ 3t \end{bmatrix} e^{4t}$

(d) $c_1 \begin{bmatrix} -3 \\ 3 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} -3t+1 \\ 2t \end{bmatrix} e^{4t}$

(e) $c_1 \begin{bmatrix} -3 \\ 3 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} 3t+1 \\ 3t \end{bmatrix} e^{4t}$

17. The characteristic equation of a matrix A is $(\lambda - 1)(\lambda + 2)^3 = 0$, where we have only two linearly independent eigenvectors corresponding to $\lambda = -2$. The Jordan normal form of A is

(a) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$

(e) $\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

18. A possible fundamental matrix for the system $X' = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix} X$ is $\Phi(t) =$

(a) $\begin{bmatrix} 1 & e^{4t} \\ 2 & -2e^{4t} \end{bmatrix}$

(b) $\begin{bmatrix} 1 & e^{4t} \\ 1 & -e^{4t} \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2e^{4t} \\ 2 & -e^{4t} \end{bmatrix}$

(d) $\begin{bmatrix} 2 & e^{4t} \\ 1 & 2e^{4t} \end{bmatrix}$

(e) $\begin{bmatrix} 1 & e^{4t} \\ 0 & -2e^{4t} \end{bmatrix}$

19. Let $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 2 \end{bmatrix}$. If $e^{At} = \begin{bmatrix} e^{2t} & f(t) & h(t) \\ 0 & e^{2t} & 6te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix}$, then $f(1) + h(1) =$

(Hint: Consider $A = D + B$, where B is a nilpotent matrix)

(a) $16e^2$

(b) $17e^2$

(c) $18e^2$

(d) $15e^2$

(e) $14e^2$

20. Let $F(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $e^{At} = \begin{bmatrix} 1+2t & -4t \\ t & 1-2t \end{bmatrix}$. A particular solution for the system $X' = AX + F(t)$ with $X_p(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ will satisfy $X_p(1) =$

(a) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(c) $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

(d) $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$