1. Which one of the following differential equations is **Exact**? check $M_y = N_x$

(a) $(y\cos x + \ln y)\,dx + \left(\sin x + \dfrac{x}{y}\right)dy = 0$ M N $M_y = \cos x + \frac{1}{y}$; $N_x = \cos x + \frac{1}{y}$ ______ (correct) ✓

(b) $(6xy - y^3)\,dx + (x^3 - 3xy^2)\,dy = 0$ $M_y = 6x - 3y^2$, $N_x = 3x^2 - 3y^2$ X

(c) $(x + \tan^{-1} y)\,dx + (y + \tan^{-1} x)\,dy = 0$ $M_y = \frac{1}{1+y^2}$, $N_x = \frac{1}{1+x^2}$ X

(d) $\left(\dfrac{y}{x} + y\right)dx + \left(\dfrac{x}{y} + x\right)dy = 0$ $M_y = \frac{1}{x} + 1$, $N_x = \frac{1}{y} + 1$ X

(e) $(xe^y + y^2)\,dx + (ye^x + 2xy)\,dy = 0$ $M_y = xe^y + 2y$, $N_x = ye^x + 2y$ X

Defn of exact eq.

~ #15 §1.2

2. A particle is moving in a straight line with acceleration

$$a(t) = 12(t+1)^2,$$

an initial position $x(0) = 4$, and an initial velocity $v(0) = 5$.
Find $x(1)$ (the position of the particle at $t = 1$).

$v(t) = \int 12(t+1)^2\,dt$
$= 4(t+1)^3 + C$

(a) 20 ______ (correct)

(b) 30

(c) 15

(d) 25

(e) 35

• $v(0) = 5 \Rightarrow 5 = 4(0+1)^3 + C \Rightarrow C = 1$

$v(t) = 4(t+1)^3 + 1$

$x(t) = \int 4(t+1)^3 + 1 \;\; dt$
$= (t+1)^4 + t + D$

• $x(0) = 4 \Rightarrow 4 = (0+1)^4 + 0 + D \Rightarrow D = 3$

$x(t) = (t+1)^4 + t + 3$

$\Rightarrow x(1) = (1+1)^4 + 1 + 3 = 16 + 4 = 20$

3. Find a general solution of the differential equation

~ #28 §1.4

$$\sqrt{x}\,\frac{dy}{dx} = 2\cos^2 y$$

a separable eq:

$$\sec^2 y \; dy = \frac{2}{\sqrt{x}}\, dx$$

$$\xRightarrow{\int} \tan y = 4\sqrt{x} + C$$

(a) $y = \tan^{-1}(4\sqrt{x} + C)$ ________________ (correct)

$$\xRightarrow{\text{or}} y = \tan^{-1}(4\sqrt{x} + C).$$

(b) $\tan\sqrt{y} = 2\sqrt{x} + C$

(c) $y = \tan^{-1}\left(\frac{2}{\sqrt{x}} + C\right)$

(d) $\tan y^2 = \sqrt{x} + C$

(e) $\cos^2 y = 3\sqrt{x} + C$

~ #6, 16 §1.1

4. Find **all** values of r for which $y = xe^{rx}$ is a solution of the differential equation $y'' + 4y' + 4y = 0$.

• $y' = x \cdot re^{rx} + e^{rx}$

• $y'' = x \cdot r^2 e^{rx} + re^{rx} + re^{rx} = r^2 x e^{rx} + 2re^{rx}$

(a) $r = -2$ ________________ (correct)

(b) $r = -2$ and $r = 2$

(c) $r = 0$ and $r = 1$

(d) $r = -1$ and $r = 1$

(e) $r = -3$

substitute in DE:

$$r^2 x e^{rx} + 2re^{rx} + 4(rxe^{rx} + e^{rx}) + 4xe^{rx} = 0$$

$$e^{rx}\left[r^2 x + 2r + 4rx + 4 + 4x\right] = 0$$

$$e^{rx}\left[(r^2 + 4r + 4)x + (2r + 4)\right] = 0$$

$$\xRightarrow{e^{rx} \neq 0} (r^2 + 4r + 4)x + (2r + 4) = 0$$

$$\Rightarrow r^2 + 4r + 4 = 0 \;\&\; 2r + 4 = 0$$

$$\Rightarrow (r+2)^2 = 0 \;\&\; 2(r+2) = 0$$

$$\Rightarrow r = -2$$

$\sim$ #43 §1.4

5. A pizza is removed from a freezer at $-25°C$, and left to rest at room temperature which is $20°C$. After 30 minutes, the temperature of the pizza is $5°C$. What is the temperature of the pizza after 60 minutes.

$T(0) = -25$, $T_m = 20$, $T(30) = 5$; $T(60) = ?$

(a) 15 ______ (correct)

(b) 20

(c) 25

(d) 18

(e) 22

$T(t) = T_m + C e^{kt}$. Find C & k?

$T(0) = -25 \Rightarrow -25 = 20 + C \Rightarrow \boxed{C = -45}$

$T(30) = 5 \Rightarrow 5 = 20 - 45 e^{30k} \Rightarrow -15 = -45 e^{30k}$

$\Rightarrow e^{30k} = \frac{1}{3} \Rightarrow 30k = -\ln 3 \Rightarrow \boxed{k = -\frac{\ln 3}{30}}$

$\Rightarrow T(t) = 20 - 45 e^{\left(-\frac{\ln 3}{30}\right)t}$

$\Rightarrow T(60) = 20 - 45 e^{-2\ln 3}$

$= 20 - 45 \cdot 3^{-2}$

$= 20 - \frac{45}{9} = 20 - 5 = 15.$

$\sim$ #29(b) §1.4

6. A general solution of the differential equation $y' = 3\sqrt{xy}$ is $y = (x^{\frac{3}{2}} + C)^2$. Which one of the following statements is **True**?

(a) $y = 0$ is a singular solution There is no C that will give $y=0$ ______ (correct)

(b) $y = 1$ is a singular solution $y=1$ is not a solution

(c) $y = x^3$ is a singular solution $C=0 \Rightarrow y = x^3$

(d) $y = (x^{3/2} - 1)^2$ is a singular solution $C=-1 \Rightarrow y = (x^{3/2}-1)^2$

(e) there is no singular solution $y=0$ is a singular solution

$\sim$ #19 §1.5

7. Solve the initial-value problem

$$y' - y\tan x = \sin x, \quad y\left(\frac{\pi}{4}\right) = \sqrt{2}$$

(assume $x \in (0, \frac{\pi}{2})$)

• $\mu(x) = e^{\int -\tan x \, dx} = e^{-\ln|\sec x|} = e^{\ln|\cos x|}$

$= |\cos x| = \cos x, \quad x \in (0, \pi/2)$

• $\cos x \, y' - y \sin x = \cos x \sin x$

$\Rightarrow \frac{d}{dx}[\cos x \cdot y] = \cos x \sin x$

$\Rightarrow \cos x \cdot y = -\frac{1}{2}\cos^2 x + C$

$\Rightarrow y = -\frac{1}{2}\cos x + C\sec x$

(a) $y = -\dfrac{1}{2}\cos x + \dfrac{5}{4}\sec x$ ______ (correct)

(b) $y = \dfrac{1}{2}\cos x + \dfrac{3}{4}\sec x$

(c) $y = -\cos x + \dfrac{3}{2}\sec x$

(d) $y = -\dfrac{1}{2}\cos x \cdot \cot x + \dfrac{5}{4}\csc x$

(e) $y = \dfrac{1}{2}\cos x \cdot \cot x + \dfrac{3}{4}\csc x$

• $y(\frac{\pi}{4}) = \sqrt{2} \Rightarrow \sqrt{2} = -\frac{1}{2}\cdot\frac{\sqrt{2}}{2} + C\cdot\sqrt{2}$

$\Rightarrow \frac{5}{4}\sqrt{2} = C\cdot\sqrt{2}$

$\Rightarrow C = \frac{5}{4}$

Sol is $y = -\frac{1}{2}\cos x + \frac{5}{4}\sec x$

#17, Ch 1 Review problems p. 88

8. Find a general solution of the **exact** differential equation

$$\underbrace{(e^x + ye^{xy})}_{M}\,dx + \underbrace{(e^y + xe^{yx})}_{N}\,dy = 0$$

There is $F(x,y)$ s.t. $F_x = M$ & $F_y = N$

• $F_x = M \Rightarrow F_x = e^x + y e^{xy}$

$\xrightarrow{\int \cdot\, dx} F(x,y) = e^x + e^{xy} + g(y)$

(a) $e^x + e^y + e^{xy} = C$ ______ (correct)

(b) $ye^x + xe^y = C$

(c) $e^{xy} - e^x - e^y = C$

(d) $e^x + e^y + 2e^{xy} = C$

(e) $e^x + e^y + xye^{xy} = C$

• $F_y = N \Rightarrow 0 + xe^{xy} + g'(y) = e^y + xe^{yx}$

$\Rightarrow g'(y) = e^y$

$\Rightarrow g(y) = e^y$ (no need for constant of integration)

Now $F(x,y) = e^x + e^{xy} + e^y$

& the sol is

$F(x,y) = C$

i.e. $e^x + e^{xy} + e^y = C$.

~ #47
§1.6

9. Find a general solution of the differential equation

$$y'' = 2(y')^2$$

(Assume $y' > 0$)

Let $u = y'$. Then $u' = y''$
So $y'' = 2(y')^2$ becomes
$u' = 2u^2$, a sep. eq.
$\frac{1}{u^2}\,du = 2\,dx$

(a) $y = B - \frac{1}{2}\ln|2x + A|$ ______ (correct)

(b) $y = B + \frac{1}{2}\ln|2x + A|$

(c) $y = B - \ln|2x + A|$

(d) $y = B + 2\ln|2x + A|$

(e) $y = B - 2\ln|2x + A|$

$\Rightarrow -\frac{1}{u} = 2x + A$
$\Rightarrow u = -\frac{1}{2x+A}$
$\Rightarrow y' = -\frac{1}{2x+A}$
$\Rightarrow y = -\frac{1}{2}\ln|2x+A| + B.$

Example 5, §1.6

10. By making a suitable substitution, the differential equation

$$x\frac{dy}{dx} + 6y = 3xy^{4/3}$$

$\Rightarrow \frac{dy}{dx} + \frac{6}{x}y = 3y^{4/3}$, a Bernoulli eq.

can be transformed into the differential equation

Let $v = y^{1-4/3} \Rightarrow v = y^{-1/3}$
$\Rightarrow y = v^{-3}$
$\Rightarrow \frac{dy}{dx} = -3v^{-4}\frac{dv}{dx}$

(a) $\dfrac{dv}{dx} - \dfrac{2}{x}v = -1$ ______ (correct)

(b) $\dfrac{dv}{dx} - \dfrac{3}{x}v = -1$

(c) $\dfrac{dv}{dx} - \dfrac{6}{x}v = -x$

(d) $\dfrac{dv}{dx} - 2v = x$

(e) $\dfrac{dv}{dx} - \dfrac{2}{x}v = -2$

$-3v^{-4}\frac{dv}{dx} + \frac{6}{x}v^{-3} = 3(v^{-3})^{4/3} = 3v^{-4}$

$\Rightarrow \frac{dv}{dx} - \frac{2}{x}v = -1$

11. Solve the differential equation

~ Example 1 §1.6

$$\frac{dy}{dx} = (2x+y)^2 - 2$$

Let $u = 2x+y$. Then $\frac{du}{dx} = 2 + \frac{dy}{dx}$

$\Rightarrow \frac{du}{dx} - 2 = u^2 - 2 \Rightarrow \frac{du}{dx} = u^2$, a sep. eq.

$\Rightarrow \frac{1}{u^2}\,du = dx$

$\overset{\int}{\Rightarrow} -\frac{1}{u} = x + C \Rightarrow u = -\frac{1}{x+C}$

(a) $y = -\dfrac{2x^2+2Cx+1}{x+C}$ ______ (correct)

(b) $y = \dfrac{2x^2+2x+1}{x+1} + C$

(c) $y = \dfrac{2x^2+x+1}{x+C}$

(d) $y = -x\ln|x+C|$

(e) $y = -\dfrac{x^2+Cx+2}{x+C}$

$\Rightarrow 2x+y = -\frac{1}{x+C}$

$\Rightarrow y = -\frac{1}{x+C} - 2x$

$= -\frac{1+2x(x+C)}{x+C}$

$\Rightarrow y = -\frac{1+2x^2+2Cx}{x+C}$

~ #12 §4.1

12. Let $u = (4,1)$, $v = (-2,-1)$, $w = (-3,1)$ be vectors in $\mathbb{R}^2$. If $w = au + bv$, then $a + b =$

$w = au + bv \Rightarrow (-3,1) = a(4,1) + b(-2,-1)$

$= (4a, a) + (-2b, -b)$

(a) -6 ______ (correct)

(b) 0

(c) $-\dfrac{3}{2}$

(d) -4

(e) 1

$= (4a-2b,\ a-b)$

$\Rightarrow \begin{cases} 4a - 2b = -3 & \text{— (1)} \\ a - b = 1 & \text{— (2)} \end{cases}$

$(2) \Rightarrow a = b+1$ — (3)

$\overset{(1)}{\Rightarrow} 4b + 4 - 2b = -3 \Rightarrow 2b = -7 \Rightarrow b = -\frac{7}{2}$

$\overset{(3)}{\Rightarrow} a = -\frac{7}{2} + 1 = \frac{-5}{2}$

So $a + b = -\frac{5}{2} - \frac{7}{2} = -\frac{12}{2} = -6$

13. Find all values of k for which the vectors of $\mathbb{R}^3$

$\sim$ # 16
§ 4.1

$$u = (3, -4, 5),\ v = (1, 1, 0), w = (k, k+1, -k)$$

are linearly independent.

L. indep $\iff \begin{vmatrix} u & v & w \end{vmatrix} \neq 0$

(a) $k \neq \frac{5}{7}$ (correct)

(b) $k \neq 3$

(c) $k \neq 1$

(d) $k \neq \frac{3}{7}$

(e) $k \neq \frac{1}{2}$

$\iff \begin{vmatrix} 3 & 1 & k \\ -4 & 1 & k+1 \\ 5 & 0 & -k \end{vmatrix} \neq 0$

$\iff 3(-k) - (4k - 5k - 5) + k(-5) \neq 0$

$\iff -3k + k + 5 - 5k \neq 0$

$\iff -7k + 5 \neq 0$

$\iff k \neq \frac{5}{7}$

$\sim$ # 1 → 14 in § 4.2

14. Which one of the following statements is **True** about the following subsets of $\mathbb{R}^4$.

$$U = \{(x_1, x_2, x_3, x_4) : x_2 x_3 = 0\}$$
$$V = \{(x_1, x_2, x_3, x_4) : x_1 = 1\}$$
$$W = \{(x_1, x_2, x_3, x_4) : x_4 = 2x_1\}$$

→ $(1,1,0,1),\ (1,0,1,0) \in U$

But $(1,1,0,1) + (1,0,1,0) = (2,1,1,2) \notin U$

since $x_2 x_3 = (1)(1) \neq 0$

So U is not a subspace of $\mathbb{R}^4$

→ $(1,0,0,0),\ (1,1,1,1) \in V$

But $(1,0,0,0) + (1,1,1,1) = (2,1,1,1) \notin V$

since $x_1 = 2 \neq 1$

So V is not a subspace of $\mathbb{R}^4$.

→ W is a subspace of $\mathbb{R}^4$. We leave it to you to check.

(a) W is a subspace of $\mathbb{R}^4$ (correct)

(b) U is a subspace of $\mathbb{R}^4$

(c) V is a subspace of $\mathbb{R}^4$

(d) U, V and W are subspaces of $\mathbb{R}^4$

(e) none of U, V, and W is a subspace of $\mathbb{R}^4$

Answer KEY

Q	MASTER	CODE01	CODE02	CODE03	CODE04
1	A	D $_{2}$	A $_{2}$	E $_{1}$	D $_{2}$
2	A	A $_{1}$	E $_{1}$	C $_{4}$	E $_{5}$
3	A	C $_{4}$	E $_{4}$	E $_{2}$	D $_{3}$
4	A	A $_{3}$	B $_{5}$	C $_{3}$	E $_{4}$
5	A	B $_{5}$	C $_{3}$	B $_{5}$	E $_{1}$
6	A	B $_{6}$	C $_{11}$	A $_{10}$	C $_{10}$
7	A	B $_{11}$	D $_{8}$	A $_{6}$	A $_{9}$
8	A	E $_{7}$	C $_{10}$	D $_{7}$	A $_{11}$
9	A	C $_{9}$	B $_{6}$	E $_{8}$	D $_{7}$
10	A	A $_{10}$	D $_{9}$	B $_{11}$	B $_{6}$
11	A	E $_{8}$	D $_{7}$	D $_{9}$	E $_{8}$
12	A	C $_{12}$	A $_{13}$	E $_{13}$	E $_{12}$
13	A	B $_{14}$	B $_{14}$	B $_{12}$	D $_{13}$
14	A	C $_{13}$	B $_{12}$	C $_{14}$	E $_{14}$