§4.4
1→8

1. Find all values of k for which the following vectors

$$v_1 = (1, 7, -3),\ v_2 = (2, k, 4),\ v_3 = (k, 0, k)$$

form a basis for $\mathbb{R}^3$. They form a basis $\Leftrightarrow \begin{vmatrix} 1 & 2 & k \\ 7 & k & 0 \\ -3 & 4 & k \end{vmatrix} \neq 0$

(a) $k \neq 0$ and $k \neq -\frac{7}{2}$ ______ (correct)

$1(k^2) - 2(7k) + k(28+3k) \neq 0$

(b) $k \neq 2$

$k^2 - 14k + 28k + 3k^2 \neq 0$

(c) $k \neq 0$ and $k \neq -3$

$4k^2 + 14k \neq 0$

(d) $k \neq 1$ and $k \neq -4$

$2k(2k+7) \neq 0$

(e) $k \neq -1$

$\Rightarrow k \neq 0,\ k \neq -\frac{7}{2}$

§4.3
1→8

2. Which pairs of vectors in $\mathbb{R}^3$ are linearly independent?

(a) $v_1 = (2, -1, 3)$, $v_2 = (2, 1, -3)$ $V_2 \neq kV_1$ for any k. ______ (correct)

(b) $v_1 = (2, -1, 3)$, $v_2 = (4, -2, 6)$ $V_2 = 2V_1$, L. Dep.

(c) $v_1 = (2, -1, 3)$, $v_2 = \left(\frac{1}{3}, -\frac{1}{6}, \frac{1}{2}\right)$ $V_2 = \frac{1}{6}V_1$, L. Dep.

(d) $v_1 = (2, -1, 3)$, $v_2 = (0, 0, 0)$ $V_2 = 0\,V_1$, L. Dep.

(e) $v_1 = (2, -1, 3)$, $v_2 = (-10, 5, -15)$ $V_2 = -5V_1$, L. Dep.

3. Find the rank of the matrix

~ #11
§4.5

$$A = \begin{bmatrix} 1 & 1 & 3 & 3 & 1 \\ 2 & 3 & 7 & 8 & 3 \\ 4 & 6 & 14 & 16 & 6 \\ 3 & 1 & 7 & 5 & 4 \end{bmatrix} \xrightarrow[R_4 \to -3R_1 + R_4]{\substack{R_2 \to -2R_1 + R_2 \\ R_3 \to -4R_1 + R_3}} \begin{bmatrix} 1 & 1 & 3 & 3 & 1 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 2 & 2 & 4 & 2 \\ 0 & -2 & -2 & -4 & 1 \end{bmatrix}$$

$$\xrightarrow[R_4 \to 2R_2 + R_4]{R_3 \to -2R_2 + R_3} \begin{bmatrix} 1 & 1 & 3 & 3 & 1 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

(a) 3 ________ (correct)

(b) 4

(c) 5

(d) 1

(e) 2

$$\xrightarrow[R_3 \leftrightarrow R_4]{R_4 \to \frac{1}{4} R_4} \begin{bmatrix} 1 & 1 & 3 & 3 & 1 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

rank(A) = # of nonzero rows = 3

4. Which one of the following statements is **True** about the following vectors in $\mathbb{R}^4$:

~ #18, 19
§4.3

$v_1 = (3, 0, -4, 1)$, $v_2(-3, -6, 4, 2)$, $v_3 = (-3, 2, 4, -2)$

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0 \Rightarrow \begin{bmatrix} 3 & -3 & -3 \\ 0 & -6 & 2 \\ -4 & 4 & 4 \\ 1 & 2 & -2 \end{bmatrix} \xrightarrow[R_3 \to -\frac{1}{4} R_3]{R_1 \to \frac{1}{3} R_1}$$

(a) v_1, v_2, v_3 are linearly dependent ________ (correct)

(b) v_1, v_2, v_3 are linearly independent

(c) $v_3 = v_1 + v_2$

(d) $v_3 = 5v_1 - v_2$

(e) $v_3 = 2v_1 + v_2$

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & -6 & 2 \\ 1 & -1 & -1 \\ 1 & 2 & -2 \end{bmatrix} \xrightarrow[\substack{R_3 \to -R_1 + R_3 \\ R_4 \to -R_1 + R_4}]{R_2 \to -\frac{1}{6} R_2}$$

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \\ 0 & 3 & -1 \end{bmatrix} \xrightarrow{R_4 \to -3R_2 + R_3}$$

Taking $c_3 = 3 \Rightarrow c_1 = 4,\ c_2 = 1$
we get

$4v_1 + v_2 + 3v_3 = 0$

$\Rightarrow$ L. dep.

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$c_2 = \frac{1}{3} c_3$

$c_1 = c_2 + c_3 = \frac{4}{3} c_3$

c_3 arbitrary

infinitely many sol.

#19
§4.4

5. Find a basis for the solution space of the system

$$\begin{cases} x_1 - 3x_2 - 9x_3 - 5x_4 = 0 \\ 2x_1 + x_2 - 4x_3 + 11x_4 = 0 \\ x_1 + 3x_2 + 3x_3 + 13x_4 = 0 \end{cases}$$

$$\left[\begin{array}{cccc|c} 1 & -3 & -9 & -5 & 0 \\ 2 & 1 & -4 & 11 & 0 \\ 1 & 3 & 3 & 13 & 0 \end{array}\right] \begin{array}{l} R_2 \to -2R_1 + R_2 \\ R_3 \to -R_1 + R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -3 & -9 & -5 & 0 \\ 0 & 7 & 14 & 21 & 0 \\ 0 & 6 & 12 & 18 & 0 \end{array}\right] \begin{array}{l} R_2 \to \frac{1}{7} R_2 \\ R_3 \to \frac{1}{6} R_3 \end{array}$$

(a) $\{(3,-2,1,0),\ (-4,-3,0,1)\}$ ________________ (correct)

(b) $\{(-1,-5,1,1)\}$

(c) $\{(6,-4,2,0)\}$

(d) $\{(4,3,0,-1),\ (-20,-15,0,5)\}$

(e) $\{(0,2,-1,1),\ (3,1,0,4)\}$

$$\left[\begin{array}{cccc|c} 1 & -3 & -9 & -5 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{array}\right] R_3 \to -R_2 + R_3$$

$$\left[\begin{array}{cccc|c} 1 & -3 & -9 & -5 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = t,\ x_4 = s$

$x_2 = -2t - 3s$

$x_1 = 3x_2 + 9x_3 + 5x_4$

$= -6t - 9s + 9t + 5s$

$= 3t - 4s$

$t, s \in \mathbb{R}$

$(x_1, x_2, x_3, x_4) = (3t - 4s, -2t - 3s, t, s)$

$= t\,\underbrace{(3,-2,1,0)} + s\,\underbrace{(-4,-3,0,1)}$

Basis $= \{(3,-2,1,0), (-4,-3,0,1)\}$

#8
§5.2

6. Find the Wronskian of the functions $f(x) = e^x$, $g(x) = e^{2x}$, $h(x) = e^{3x}$ on $(-\infty, \infty)$.

(a) $2e^{6x}$ ________________ (correct)

(b) $6e^{6x}$

(c) $18e^{6x}$

(d) $-4e^{6x}$

(e) $-8e^{6x}$

$$W(f,g,h) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix}$$

$= e^x(18e^{5x} - 12e^{5x}) - e^{2x}(9e^{4x} - 3e^{4x}) + e^{3x}(4e^{3x} - 2e^{3x})$

$= e^x \cdot 6e^{5x} - e^{2x} \cdot 6e^{4x} + e^{3x} \cdot 2e^{3x}$

$= 6e^{6x} - 6e^{6x} + 2e^{6x}$

$= 2e^{6x}$

§5.2
1→7

7. Which set of the following functions forms a **linearly independent** set of functions on $(-\infty, \infty)$.

(a) $\{1, 2x, x^3\}$ none is a linear combination of the other two. (correct)

(b) $\{0, x, e^x\}$ $2\cdot 0 + 0\cdot x + 0\cdot e^x = 0$ L. Dep.

(c) $\{x, 2x^2, 3x - x^2\}$ $3x - x^2 = 3(x) - \frac{1}{2}(2x^2)$, L. Dep.

(d) $\{2, \sin^2 x, \cos^2 x\}$ $2 = 2\cdot\sin^2 x + 2\cdot\cos^2 x$, L. Dep.

(e) $\{1, \cos^2 x, \cos(2x)\}$ $\cos(2x) = 2\cdot\cos^2 x - 1$, L. Dep.

#12
§5.1

8. If $y = Ae^{4x}\cos(3x) + Be^{4x}\sin(3x)$ is the solution of the initial value problem

$$y'' - 8y' + 25y = 0,\ y(0) = 3,\ y'(0) = 0,$$

then $AB =$

- $y(0) = 3 \Rightarrow 3 = A + 0 \Rightarrow \boxed{A = 3}$
- $y' = Ae^{4x}\cdot -3\sin(3x) + \cos(3x)\cdot 4Ae^{4x}$
$+ Be^{4x}\cdot 3\cos(3x) + \sin(3x)\cdot 4Be^{4x}$

$y'(0) = 0 \Rightarrow 0 = 0 + 4A + 3B + 0$
$\Rightarrow 0 = 12 + 3B \Rightarrow \boxed{B = -4}$

So $AB = -12$

(a) -12 (correct)

(b) -9

(c) 0

(d) 6

(e) 10

~ Example 2 §5.3

9. Find a general solution of the differential equation

$$y^{(5)} + y^{(4)} - 2y^{(3)} = 0$$

ch. eq: $r^5 + r^4 - 2r^3 = 0$

$\Rightarrow r^3(r^2+r-2) = 0$

$\Rightarrow r^3(r+2)(r-1) = 0$

$\Rightarrow r = 0$ (rep. 3 times), $r = -2$, $r = 1$

(a) $y = c_1 + c_2x + c_3x^2 + c_4e^x + c_5e^{-2x}$ (correct)

$1, x, x^2$ e^{-2x} e^x

(b) $y = c_1 + c_2x + c_3x^2 + c_4e^x + c_5xe^x$

(c) $y = c_1 + c_2x + c_3x^2 + c_4e^{-x} + c_5e^{2x}$

$y = c_1 + c_2x + c_3x^2 + c_4e^{-2x} + c_5e^x$

(d) $y = c_1 + c_2e^x + c_3e^{-2x}$

(e) $y = c_1 + c_2x + c_3e^x + c_4e^{2x}$

#28 §5.3

10. Let r_1, r_2, r_3 be the roots of the characteristic equation of the differential equation

$$3y''' - 7y'' - 7y' + 3y = 0.$$

ch. eq: $3r^3 - 7r^2 - 7r + 3 = 0$

If $r_1 \le r_2 \le r_3$, then $r_1^2 + r_2r_3 =$

by inspection, $r = -1$ is a root

$$\begin{array}{r|l} & 3r^2 - 10r + 3 \\ \hline r+1 & 3r^3 - 7r^2 - 7r + 3 \\ & 3r^3 + 3r^2 \\ \hline & -10r^2 - 7r + 3 \\ & -10r^2 - 10r \\ \hline & 3r + 3 \\ & 3r + 3 \\ \hline & 0 \end{array}$$

(a) 2 (correct)

(b) 0

(c) $\frac{10}{9}$

(d) $-\frac{5}{3}$

(e) $\frac{7}{3}$

$(r+1)(3r^2 - 10r + 3) = 0$

$(r+1)(3r-1)(r-3) = 0$

$\Rightarrow r = -1, \frac{1}{3}, 3$

$r_1 = -1, \ r_2 = \frac{1}{3}, \ r_3 = 3$ $(r_1 \le r_2 \le r_3)$

$\Rightarrow r_1^2 + r_2r_3 = 1 + 1 = 2$

§5.3
39 → 42

11. Find a differential equation with constant coefficients whose general solution is

$$y = c_1e^{3x} + c_2e^{-x}\sin x + c_3e^{-x}\cos x$$

roots of ch. eq: $3, -1+i, -1-i$

Factors: $r-3, r+1-i, r+1+i$

(a) $y''' - y'' - 4y' - 6y = 0$ ____ (correct)

(b) $y''' - 2y'' - y' - 4y = 0$

(c) $y''' + y'' + 4y' - 6y = 0$

(d) $y''' + 3y'' - 4y' + 6y = 0$

(e) $y''' - y'' - 6y' = 0$

Eq: $(r-3)(r+1-i)(r+1+i) = 0$

$(r-3)((r+1)^2 - i^2) = 0$

$(r-3)(r^2+2r+2) = 0$

$r^3 + 2r^2 + 2r$
$- 3r^2 - 6r - 6 = 0$

$\Rightarrow r^3 - r^2 - 4r - 6 = 0$

$\Rightarrow y''' - y'' - 4y' - 6y = 0$

~ #29, 30
§5.5

12. An appropriate form of a particular solution of the differential equation

$$(D+1)(D-1)^2(D^2+4)y = 1 - 3e^x + 2x\,\sin(2x)$$

is $r=-1$, 1 (twice), $r = \pm 2i$

(a) $y_p = A + Bx^2e^x + (Cx + Dx^2)\cos(2x) + (Ex + Fx^2)\sin(2x)$ ____ (correct)

(b) $y_p = A + Be^x + (Cx + D)\cos(2x) + (Ex + F)\sin(2x)$

(c) $y_p = A + Bx^2e^x + (Cx + D)\sin(2x)$

(d) $y_p = A + Bxe^x + (Cx + Dx^2)\sin(2x)$

(e) $y_p = A + Bxe^x + (Cx + Dx^2)\cos(2x) + (Ex + Fx^2)\sin(2x)$

$y_c = c_1e^{-x} + c_2e^x + c_3xe^x + c_4\cos(2x) + c_5\sin(2x)$

1st choice of y_p: $1 - 3e^x + 2x\sin(2x) \Rightarrow y_p = A + Be^x + (Cx+D)\cos(2x) + (Ex+F)\sin(2x)$

To avoid repetition with y_c, we take

$$y_p = A + Bx^2e^x + (Cx^2+Dx)\cos(2x) + (Ex^2+Fx)\sin(2x)$$

13. If y_p is a particular solution of the differential equation

~ Examples §5.5

$$y'' + 3y' - 4y = 2x^2$$

: $r^2+3r-4=0 \Rightarrow (r+4)(r-1)=0 \Rightarrow r=-4, r=1$

then $y_p(-1) =$

$y_c = c_1 e^{-4x} + c_2 e^{x}$] no duplicate

$y_p = Ax^2 + Bx + C$

$y_p' = 2Ax + B$

$y_p'' = 2A$

(a) $-\dfrac{9}{16}$ ______ (correct)

(b) $-\dfrac{1}{2}$

(c) $\dfrac{7}{4}$

(d) $\dfrac{5}{16}$

(e) $\dfrac{3}{4}$

$y_p'' + 3y_p' - 4y_p = 2x^2$

$2A + 6Ax + 3B - 4Ax^2 - 4Bx - 4C = 2x^2$

$\Rightarrow -4A = 2 \;;\; 6A - 4B = 0 \;;\; 2A + 3B - 4C = 0$

$\Rightarrow A = -\frac{1}{2} \;;\; B = -\frac{3}{4} \;;\; C = -\frac{13}{16}$

So $y_p = -\frac{1}{2}x^2 - \frac{3}{4}x - \frac{13}{16}$

$y_p(-1) = -\frac{1}{2} + \frac{3}{4} - \frac{13}{16} = \frac{1}{4} - \frac{13}{16} = \frac{4-13}{16} = \frac{-9}{16}$

~ #54 §5.5

14. A particular solution of the differential equation

$$y'' + 16y = 16\sec^2(4x)$$

: $r^2 + 16 = 0 \Rightarrow r = \pm 4i$

$y_c = c_1 \underbrace{\cos(4x)}_{y_1} + c_2 \underbrace{\sin(4x)}_{y_2}$

is $f(x) = 16\sec^2(4x)$

$y_p = u_1(x)\cos(4x) + u_2(x)\sin(4x)$

(a) $y_p = -1 + \sin(4x) \cdot \ln|\sec(4x) + \tan(4x)|$ ______ (correct)

(b) $y_p = -1 + \cos(4x) \cdot \ln|\sec(4x) + \tan(4x)|$

(c) $y_p = -\tan(4x) + \sec(4x)\tan(4x)$

(d) $y_p = -4 + \sin(4x) \cdot \ln|\sec(4x) + \tan(4x)|$

(e) $y_p = -\sec(4x) + \ln|\sec(4x) + \tan(4x)|$

$W(\cos 4x, \sin 4x) = \begin{vmatrix} \cos(4x) & \sin(4x) \\ -4\sin(4x) & 4\cos 4x \end{vmatrix} = 4$

$u_1'(x) = -\dfrac{y_2 f(x)}{W} = -\dfrac{\sin(4x) \cdot 16\sec^2(4x)}{4} = -4\tan(4x)\sec(4x) \Rightarrow u_1(x) = -\sec(4x)$

$u_2'(x) = \dfrac{y_1 f(x)}{W} = \dfrac{\cos(4x) \cdot 16\sec^2(4x)}{4} = 4\sec(4x) \Rightarrow u_2(x) = \ln|\sec(4x) + \tan(4x)|$

So $y_p(x) = -\sec(4x) \cdot \cos(4x) + \ln|\sec(4x) + \tan(4x)| \cdot \sin(4x)$

$= -1 + \sin(4x) \cdot \ln|\sec(4x) + \tan(4x)|$.

Answer KEY

Q	MASTER	CODE01	CODE02	CODE03	CODE04
1	A	E $_4$	A $_5$	C $_3$	B $_5$
2	A	E $_3$	B $_4$	D $_1$	A $_1$
3	A	A $_2$	B $_3$	D $_4$	C $_3$
4	A	A $_5$	B $_1$	C $_5$	A $_2$
5	A	B $_1$	B $_2$	D $_2$	E $_4$
6	A	E $_9$	B $_9$	E $_6$	D $_6$
7	A	D $_8$	C $_8$	D $_9$	E $_7$
8	A	A $_7$	C $_7$	D $_7$	E $_8$
9	A	E $_6$	A $_6$	C $_8$	D $_9$
10	A	A $_{12}$	D $_{11}$	A $_{14}$	B $_{12}$
11	A	B $_{13}$	D $_{12}$	E $_{10}$	A $_{10}$
12	A	D $_{11}$	A $_{13}$	A $_{11}$	D $_{11}$
13	A	A $_{10}$	D $_{10}$	B $_{13}$	D $_{14}$
14	A	B $_{14}$	B $_{14}$	A $_{12}$	A $_{13}$