~#2
§ 1.4

1. If $y(x)$ is the solution of the initial value problem

$$y' + e^x y^2 = 0,\ y(0) = \frac{1}{2},$$

then $y(\ln 2) =$

sep. $y' = -e^x y^2 \Rightarrow dy = -e^x y^2\, dx$

$\Rightarrow y^{-2} \cdot dy = -e^x\, dx$

$\Rightarrow -y^{-1} = -e^x + C$

$\Rightarrow -\frac{1}{y} = -e^x + C$

$\Rightarrow y = \frac{-1}{-e^x + C}$

$y(0) = \frac{1}{2} \Rightarrow \frac{1}{2} = \frac{-1}{-1+C} \Rightarrow -1 + C = -2 \Rightarrow C = -1$

$y = \frac{-1}{-e^x - 1} = \frac{1}{e^x + 1}$

$y(\ln 2) = \frac{1}{2+1} = \frac{1}{3}$

(a) $\frac{1}{3}$ ________________ (correct)

(b) $\frac{2}{3}$

(c) $\frac{1}{4}$

(d) $\frac{1}{2}$

(e) $\frac{3}{4}$

~#34
§ 1.4

2. In a certain culture of bacteria, the number of bacteria doubled after 4 hours. How long did it take for the population to triple.
(Assume the rate of change of population is proportional to the population present at time t).

$P(t) = C e^{kt}$

. $P(0) = P_0 \Rightarrow C = P_0$

. $P(4) = 2P_0 \Rightarrow 2P_0 = P_0 e^{k4}$

$\Rightarrow 2 = e^{k4} \Rightarrow \ln 2 = k4 \Rightarrow k = \frac{\ln 2}{4}$

. $P(t) = 3P_0 \Rightarrow P_0 e^{kt} = 3P_0$

$\Rightarrow e^{kt} = 3$

$\Rightarrow kt = \ln 3$

$\Rightarrow t = \frac{\ln 3}{k} = \frac{\ln 3}{\frac{\ln 2}{4}} = \frac{4 \ln 3}{\ln 2} = \frac{\ln 81}{\ln 2}$

(a) $\frac{\ln 81}{\ln 2}$ ________________ (correct)

(b) $\frac{\ln 27}{\ln 2}$

(c) $\frac{\ln 9}{\ln 2}$

(d) $\frac{\ln 3}{\ln 2}$

(e) $\frac{1}{\ln 2}$

Example 2
§1.6

3. Solve the **homogeneous** differential equation

$$2xy\frac{dy}{dx} = 4x^2 + 3y^2.$$

$\frac{dy}{dx} = \frac{4x^2+3y^2}{2xy} = 2\left(\frac{x}{y}\right) + \frac{3}{2}\left(\frac{y}{x}\right)$

Let $v = \frac{y}{x}$. Then $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

(a) $y^2 + 4x^2 = kx^3$ ____________ (correct)

(b) $y^2 - 2x^2 = kx^3$

(c) $3y^2 + x^3 = kx^4$

(d) $-5y^2 + x^2 = kx^3$

(e) $y^3 - x^3 = x^2 + k$

$v + x\frac{dv}{dx} = 2\frac{1}{v} + \frac{3}{2}v \Rightarrow x\frac{dv}{dx} = \frac{2}{v} + \frac{1}{2}v = \frac{4+v^2}{2v}$

$\Rightarrow \frac{2v}{v^2+4}\,dv = \frac{1}{x}\,dx$

$\Rightarrow \ln(v^2+4) = \ln|x| + \ln|c| = \ln|xc|$

$\Rightarrow v^2+4 = |cx| = \pm cx = kx$

$\Rightarrow \frac{y^2}{x^2} + 4 = kx \Rightarrow y^2 + 4x^2 + kx^3.$

Example 4
§4.1

4. Let $t = (4, 20, 23)$, $u = (1, 3, 2)$, $v = (2, 8, 7)$, $w = (1, 7, 9)$ be vectors in $\mathbb{R}^3$. If $t = au + bv + cw$, then $a + b + c =$

(a) 6 ____________ (correct)

(b) 5

(c) 4

(d) -7

(e) 8

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{array}\right] \xrightarrow[R_3 \to -2R_1+R_2]{R_2 \to -3R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 2 & 4 & 8 \\ 0 & 3 & 7 & 15 \end{array}\right]$$

$$\xrightarrow{R_2 \to \frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 3 & 7 & 15 \end{array}\right] \xrightarrow{R_3 \to -3R_2+R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$\Rightarrow c = 3$

$b + 2c = 4 \Rightarrow b + 6 = 4 \Rightarrow b = -2$

$a + 2b + c = 4 \Rightarrow a - 4 + 3 = 4 \Rightarrow a = 5$

$a + b + c = 5 - 2 + 3 = 6$

~ #5
§ 4.5

5. If the rank of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & k \\ 1 & 2 & 2 & 1 \\ 1 & 2 & k & 1 \end{bmatrix} \xrightarrow[R_3 \to -R_1 + R_3]{R_2 \to -R_1 + R_2} \begin{bmatrix} 1 & 1 & 1 & k \\ 0 & 1 & 1 & 1-k \\ 0 & 1 & k-1 & 1-k \end{bmatrix}$$

is equal to 2, then $k =$

$$\xrightarrow{R_3 \to -R_2 + R_3} \begin{bmatrix} 1 & 1 & 1 & k \\ 0 & 1 & 1 & 1-k \\ 0 & 0 & k-2 & 0 \end{bmatrix}$$

(a) 2 ____(correct)

(b) −2

(c) 1

(d) −1

(e) 0

$\text{Rank} = 2 \iff k - 2 = 0$

$\iff k = 2$

§ 4.5
Relation between rank & dim of sol space

6. Let A be a 10×15 matrix. If the rank of A is 7, then the dimension of the solution space of the system $AX = 0$ is equal to

$15 - 7 = 8$

(a) 8 ____(correct)

(b) 3

(c) 10

(d) 15

(e) 7

~#12
§4.4

7. The dimension of the subspace

$$W = \{(x_1, x_2, x_3, x_4, x_5) : x_1 - x_2 = x_3 + 2x_4 - 3x_5\}$$

of $\mathbb{R}^5$ is equal to

It is the solution space of the system

$x_1 - x_2 - x_3 - 2x_4 + 3x_5 = 0$

$[1 \ -1 \ -1 \ -2 \ 3 \mid 0]$

(a) 4 ________________ (correct)

(b) 3

(c) 2

(d) 1

(e) 5

L.V.: x_1

Free V.: x_2, x_3, x_4, x_5 (four free variables)

$\Rightarrow$ dim of W is 4

§5.3
Basic

8. The general solution of the differential equation

$$D(D-1)^2(D^2+4D+5)y = 0$$

is

$r = 0, 1, 1, \frac{-4 \pm \sqrt{16-4(5)}}{2} = -2 \pm i$

1, e^x, xe^x, $e^{-2x}\cos x, e^{-2x}\sin x$

(a) $y = c_1 + c_2e^x + c_3xe^x + c_4e^{-2x}\cos x + c_5e^{-2x}\sin x$ ✓ ________________ (correct)

(b) $y = c_1 + c_2e^x + c_3e^{-2x}\cos x + c_4e^{-2x}\sin x$

(c) $y = c_1 + c_2e^x + c_3xe^x + c_4e^x\cos(2x) + c_5e^x\sin(2x)$

(d) $y = c_1 + c_2e^x + c_3e^{-x}\cos(2x) + c_4e^{-x}\sin(2x)$

(e) $y = c_1 + c_2e^{-x} + c_3xe^{-x} + c_4e^x\cos(2x) + c_5e^x\sin(2x)$

~ #7
§5.5

9. An appropriate form of a **particular solution** of the differential equation

$$y'' - y = 2e^x - 3e^{-x}$$

is

$y'' - y = 0 \Rightarrow r^2 - 1 = 0 \Rightarrow r = \pm 1$

$y_c = c_1 e^x + c_2 e^{-x}$

$\Rightarrow y_p = Axe^x + Bxe^{-x}$

(a) $y_p = Axe^x + Bxe^{-x}$ ______________________ (correct)

(b) $y_p = Ae^x + Be^{-x}$

(c) $y_p = Ae^x + Bxe^x$

(d) $y_p = A + Be^x + Ce^{-x}$

(e) $y_p = Axe^x + Be^{-x}$

~ #22
§5.2

10. If $y(x)$ is the solution of the initial value problem

$$y'' + y = 1,\ y(0) = 2,\ y'(0) = 1,$$

then $y\left(\frac{\pi}{4}\right) =$

$y'' + y = 0 \ : \ r^2 + 1 = 0 \Rightarrow r = \pm i$

$y_c = c_1 \cos x + c_2 \sin x$

$y_p = 1$ (by inspection)

(a) $1 + \sqrt{2}$ ______________________ (correct)

(b) $2 + \sqrt{2}$

(c) $1 - 3\sqrt{2}$

(d) $2 + 3\sqrt{2}$

(e) $3 + 2\sqrt{2}$

$y = c_1 \cos x + c_2 \sin x + 1$

. $y(0) = 2 \Rightarrow c_1 + 0 + 1 = 2 \Rightarrow c_1 = 1$

. $y' = -c_1 \sin x + c_2 \cos x$

$y'(0) = 1 \Rightarrow c_2 = 1$

So $y = \cos x + \sin x + 1$

$y\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + 1 = \sqrt{2} + 1$

~ #53
§ 5.5

11. Using **variation of parameters**, the differential equation

$$y'' + 4y = 4\sec(2x)$$

has a particular solution $y_p = u_1 y_1 + u_2 y_2$. Then $u_1 + u_2 =$

(a) $2x + \ln|\cos(2x)|$ ______ (correct)

(b) $2x + \ln|\sec(2x) + \tan(2x)|$

(c) $2x + \ln|\csc(2x) + \cot(2x)|$

(d) $2x - \ln|\sin(2x)|$

(e) $4x - \ln|\cos(2x)|$

~ # 13
§ 6.1

12. The matrix

$$A = \begin{bmatrix} 2 & 0 & 2 \\ 1 & 1 & 1 \\ \frac{1}{2} & 0 & 2 \end{bmatrix}$$

has

(a) one eigenvalue of multiplicity 1 and one eigenvalue of multiplicity 2 ___ (correct)

(b) three distinct real eigenvalues

(c) one eigenvalue of multiplicity 3

(d) one real and one pair of nonreal complex eigenvalues

(e) no real eigenvalues

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 0 & 2 \\ 1 & 1-\lambda & 1 \\ \frac{1}{2} & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda)(2-\lambda) - 0 + 2\cdot\left(0 - \tfrac{1}{2}(1-\lambda)\right)$$

$$= (2-\lambda)^2(1-\lambda) - (1-\lambda)$$

$$= (1-\lambda)\left[(2-\lambda)^2 - 1\right]$$

$$= (1-\lambda)(2-\lambda-1)(2-\lambda+1)$$

$$= (1-\lambda)(1-\lambda)(3-\lambda)$$

$$= 0 \implies \lambda = 1, 1, 3$$

§6.1
Def^n of eigenvalues/ eigenvectors

13. If $v = \begin{bmatrix} a \\ 1 \\ b \end{bmatrix}$ is an eigenvector of the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

$AV = 3V \Rightarrow \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} a \\ 1 \\ b \end{bmatrix} = 3\begin{bmatrix} a \\ 1 \\ b \end{bmatrix}$

$\begin{bmatrix} 2a+b \\ a+2+b \\ a+2b \end{bmatrix} = \begin{bmatrix} 3a \\ 3 \\ 3b \end{bmatrix}$

associated with the eigenvalue $\lambda = 3$, then $4a + 2b =$

. $2a+b = 3a \Rightarrow \boxed{b = a}$

(a) 3 ______________________________ (correct)

. $a+2+b = 3 \Rightarrow 2a = 1$

(b) -1

$\Rightarrow a = \frac{1}{2}$ & $b = \frac{1}{2}$

(c) 1

(d) -3

, $4a+2b = 4a+2a = 6a = \frac{6}{2} = 3$

(e) 6

14. An eigenvector associated with the eigenvalue $\lambda = 9i$ of the matrix

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§6.1

$$A = \begin{bmatrix} 0 & -3 \\ 27 & 0 \end{bmatrix}$$

is

$(A - 9iI)\vec{V} = 0 \Rightarrow \left[\begin{array}{cc|c} -9i & -3 & 0 \\ 27 & -9i & 0 \end{array}\right]$

$\xrightarrow{R_1 \to 3iR_1} \left[\begin{array}{cc|c} 27 & -9i & 0 \\ 27 & -9i & 0 \end{array}\right]$

$\xrightarrow{R_2 \to -R_1+R_2}$

(a) $\begin{bmatrix} i \\ 3 \end{bmatrix}$ ______________________________ (correct)

$\left[\begin{array}{cc|c} 27 & -9i & 0 \\ 0 & 0 & 0 \end{array}\right]$

(b) $\begin{bmatrix} -i \\ 3 \end{bmatrix}$

$27v_1 - 9iv_2 = 0$

(c) $\begin{bmatrix} 2i \\ -1 \end{bmatrix}$

$v_1 = \frac{i}{3} v_2$

(d) $\begin{bmatrix} -3i \\ -6 \end{bmatrix}$

$V = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} i/3 \\ 1 \end{bmatrix} v_2$

(e) $\begin{bmatrix} i \\ 1 \end{bmatrix}$

$\xrightarrow{v_2 = 3} V = \begin{bmatrix} i \\ 3 \end{bmatrix}$

#19
§6.1

15. Let $A = \begin{bmatrix} 3 & 6 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. A basis for the eigenspace of A associated with the eigenvalue $\lambda = 1$ of A is

$(A-1I)\vec{v}=0 \Rightarrow \left[\begin{array}{ccc|c} 2 & 6 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

(a) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \right\}$ ______________________________ (correct)

$\Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

(b) $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$

$v_1 + 3v_2 - v_3 = 0 \Rightarrow v_1 = -3v_2 + v_3$

(c) $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -3v_2 + v_3 \\ v_2 \\ v_3 \end{bmatrix} = v_2 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

(d) $\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \right\}$

a basis $= \left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

(e) $\left\{ \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \right\}$

~#1
§6.2

16. Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. If P is a **diagonalizing matrix** such that $P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, then

$\lambda_1 = 1, \lambda_2 = 3 \Leftarrow$

$(A-\lambda_1 I)v = 0 \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 \to -R_1 + R_2} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

(a) $P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ ______________________________ (correct)

$v_1 + v_2 = 0 \Rightarrow v_1 = -v_2$

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -v_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} v_2 \xrightarrow{v_2 = 1} v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(b) $P = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$

$(A-\lambda_2 I)v = 0 \Rightarrow \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

(c) $P = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$

$-v_1 + v_2 = 0 \Rightarrow v_1 = v_2$

(d) $P = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$

$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} v_2 \xrightarrow{v_2 = 1} v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(e) $P = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$

$P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

#24
§ 7.2

17. If $X = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-3t}$ is the solution of the initial value problem

$$X' = \begin{bmatrix} 5 & -2 \\ 8 & -5 \end{bmatrix} X,\ X(0) = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} 5 \\ -4 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

then $c_1 - c_2 =$

$\Rightarrow 5 = c_1 + c_2$ — (1)
$-4 = c_1 + 4c_2$ — (2)

(2)−(1) : $-9 = 3c_2 \Rightarrow c_2 = -3$

(a) 11 ______________________________ (correct)

$\overset{(1)}{\Longrightarrow} c_1 = 8$

(b) 12
(c) 13
(d) 10
(e) 9

So $c_1 - c_2 = 8 - (-3) = 11$

#13
§7.2

18. Find a general solution of the system $X' = \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix} X$.

$|A - \lambda I| = \begin{vmatrix} 4-\lambda & 2 \\ -3 & -1-\lambda \end{vmatrix}$

(a) $X = c_1 \begin{bmatrix} 2e^t \\ -3e^t \end{bmatrix} + c_2 \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix}$ ______________________________ (correct)

$= -4 - 4\lambda + \lambda + \lambda^2 + 6$

(b) $X = c_1 \begin{bmatrix} e^t \\ -e^t \end{bmatrix} + c_2 \begin{bmatrix} 2e^{2t} \\ -e^{2t} \end{bmatrix}$

$= \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2) = 0$

$\Rightarrow \lambda = 1,\ \lambda = 2$

(c) $X = c_1 \begin{bmatrix} 2e^{-t} \\ e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} e^{-2t} \\ -e^{-2t} \end{bmatrix}$

(d) $X = c_1 \begin{bmatrix} -2e^t \\ 3e^t \end{bmatrix} + c_2 \begin{bmatrix} 2e^{2t} \\ 3e^{2t} \end{bmatrix}$

(e) $X = c_1 \begin{bmatrix} 2e^{-t} \\ -3e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} e^{-2t} \\ -e^{-2t} \end{bmatrix}$

• $\lambda = 1$: $\left[\begin{array}{cc|c} 3 & 2 & 0 \\ -3 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 3 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$3v_1 + 2v_2 = 0$

$v_1 = -\frac{2}{3} v_2$

$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} v_2 \\ v_2 \end{bmatrix} \xrightarrow{v_2 = -3} \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

$\underline{X_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix} e^t}$

• $\lambda = 2$ $\left[\begin{array}{cc|c} 2 & 2 & 0 \\ -3 & -3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$X = c_1 X_1 + c_2 X_2$

$v_1 + v_2 = 0 \Rightarrow v_1 = -v_2$

$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} v_2 \xrightarrow{v_2 = -1} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\underline{X_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}}$

19. If $\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} c_n x^{n-1} = 0$, then $\frac{c_{26}}{9c_{27}} =$

$K = n-2$ $\quad$ $k = n-1$

(a) 78 ____________________ (correct)

(b) 64

(c) 99

(d) 48

(e) 81

$$\sum_{K=0}^{\infty} (K+2)(K+1) C_{K+2} x^K - \sum_{K=0}^{\infty} C_{K+1} x^K = 0$$

$$\sum_{K=0}^{\infty} [(K+2)(K+1) C_{K+2} - C_{K+1}] x^K = 0$$

$$\Rightarrow (K+2)(K+1) C_{K+2} - C_{K+1} = 0$$

$$\Rightarrow \frac{C_{K+1}}{C_{K+2}} = (K+1)(K+2)$$

$$\xrightarrow{K=25} \frac{C_{26}}{C_{27}} = 26 \cdot 27 \Rightarrow \frac{C_{26}}{9C_{27}} = \frac{1}{9} \cdot 26 \cdot 27 = 26 \cdot 3 = 78$$

20. A possible **fundamental matrix** for the system $X' = \begin{bmatrix} 6 & 2 \\ 5 & 3 \end{bmatrix} X$ is

#2
§ 8.1

(a) $\Phi(t) = \begin{bmatrix} 2e^t & 2e^{8t} \\ -5e^t & 2e^{8t} \end{bmatrix}$ ____________________ (correct)

(b) $\Phi(t) = \begin{bmatrix} -2e^t & -e^{8t} \\ 5e^t & e^{8t} \end{bmatrix}$

(c) $\Phi(t) = \begin{bmatrix} 4e^t & 2e^{8t} \\ -10e^t & -3e^{8t} \end{bmatrix}$

(d) $\Phi(t) = \begin{bmatrix} e^t & -2e^{8t} \\ e^t & 5e^{8t} \end{bmatrix}$

(e) $\Phi(t) = \begin{bmatrix} -e^t & 6e^{8t} \\ 5e^t & 6e^{8t} \end{bmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 6-\lambda & 2 \\ 5 & 3-\lambda \end{vmatrix} = (6-\lambda)(3-\lambda) - 10$$

$$= \lambda^2 - 9\lambda + 18 - 10$$

$$= \lambda^2 - 9\lambda + 8 = (\lambda - 1)(\lambda - 8) = 0$$

$$\Rightarrow \lambda = 1, \lambda = 8$$

$\cdot\ \lambda = 1$: $\left[\begin{array}{cc|c} 5 & 2 & 0 \\ 5 & 2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 5 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$5V_1 + 2V_2 = 0 \Rightarrow V_1 = -\frac{2}{5}V_2$

$V = \begin{bmatrix} -2/5 \\ 1 \end{bmatrix} V_2 \xrightarrow{V_2 = -5} \begin{bmatrix} 2 \\ -5 \end{bmatrix}$

$\underline{X_1 = \begin{bmatrix} 2 \\ -5 \end{bmatrix} e^t}$

$\cdot\ \lambda = 8$ $\begin{bmatrix} -2 & 2 \\ 5 & -5 \end{bmatrix} \to \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$-2V_1 + 2V_2 = 0 \Rightarrow V_1 = V_2$

$\Phi(t) = [X_1 \quad X_2]$

$V = \begin{bmatrix} 1 \\ 1 \end{bmatrix} V_2 \xrightarrow{V_2 = 2} \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$\underline{X_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} e^{8t}}$

∼ #15
§ 11.2

21. Let $y = \sum_{k=0}^{\infty} c_k x^k$ be a power series solution of $y'' - x^2 y = 0$ about $x = 0$. Then the coefficients satisfy

$y' = \sum_{k=1}^{\infty} k c_k x^{k-1}, \quad y'' = \sum_{k=2} k(k-1) c_k x^{k-2}$

$y'' - x^2 y = 0$

(a) $c_{k+2} = \dfrac{c_{k-2}}{(k+1)(k+2)}$, $k \geq 2$ ____(correct)

$\sum_{k=2}^{\infty} k(k-1) c_k x^{k-2} - \sum_{k=0}^{\infty} c_k x^{k+2} = 0$

(b) $c_{k+2} = \dfrac{c_{k-1}}{k(k+1)}$, $k \geq 1$

$n = k-2 \qquad n = k+2$

(c) $c_{k+3} = \dfrac{c_{k+1}}{k(k+1)}$, $k \geq 1$

$\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n - \sum_{n=2} c_{n-2} x^n = 0$

(d) $c_{k+2} = \dfrac{c_{k-2}}{k(k-1)}$, $k \geq 2$

(e) $c_{k+2} = \dfrac{c_k}{(k+1)(k+2)}$, $k \geq 0$

$2c_2 + 6c_3 x + \sum_{n=2}^{\infty} \underbrace{[(n+1)(n+2) c_{n+2} - c_{n-2}]}_{=0} x^n = 0$

$\Rightarrow (n+1)(n+2) c_{n+2} = c_{n-2}$

$\Rightarrow c_{n+2} = \dfrac{c_{n-2}}{(n+1)(n+2)}, \quad n \geq 2$

∼ #30
§ 8.2

22. Let

$$F(t) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \; e^{At} = \begin{bmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{bmatrix}.$$

A **particular solution** for the system $X' = AX + F(t)$ is

$X_p = e^{At} \int e^{-At} F(t)\, dt$

(a) $X_p = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ____(correct)

(b) $X_p = \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}$

$= e^{A} \int \begin{bmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} dt$

(c) $X_p = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$

(d) $X_p = \begin{bmatrix} 2\cos(2t) \\ -2\sin(2t) \end{bmatrix}$

$= e^{At} \int \begin{bmatrix} 2\cos(2t) \\ -2\sin(2t) \end{bmatrix} dt$

(e) $X_p = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{bmatrix} \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}$

$= \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

#18
§7.3

23. If the general solution of the system #18, §7.3

$$X' = \begin{bmatrix} 6 & -4 & 6 \\ 1 & 6 & 1 \\ 1 & 6 & 1 \end{bmatrix} X$$

is $X = c_1 \begin{bmatrix} 1 \\ 1 \\ a \end{bmatrix} e^{8t} + c_2 \begin{bmatrix} b \\ -1 \\ -1 \end{bmatrix} e^{5t} + c_3 \begin{bmatrix} 1 \\ c \\ -1 \end{bmatrix}$, then $a + b + c =$

eigenvalues : 8, 5, 0

(a) 3 ________________ (correct)

(b) 2

(c) 4

(d) 1

(e) 0

Find corresponding eigenvectors

$\Rightarrow a = 1, \quad b = 2, \quad c = 0$

$\Rightarrow a + b + c = 3$

~ #1-8

§11.3 24. The sum of the roots of the **indicial equation** at $x = 0$ for $x^2y'' + 7xy' - 7y = 0$ is

$$y'' + \frac{7}{x} y' - \frac{7}{x^2} y = 0$$

(a) -6 ________________ (correct)

(b) 0

(c) 7

(d) -1

(e) 4

$p(x) = 7, \quad q(x) = -7$

$p_0 = p(0) = 7, \quad q_0 = q(0) = -7$

$r(r-1) + p_0 r + q_0 = 0$

$r(r-1) + 7r - 7 = 0$

$r^2 + 6r - 7 = 0$

$(r-1)(r+7) = 0$

$\Rightarrow r = 1, -7$

Sum $= 1 - 7 = -6$

25. The guaranteed radius of convergence of the power series solution of

Example 4
§ 11.2
also #2

$$(x^2 + 4)y'' + 4xy' + 2y = 0$$

about the ordinary point $x = 0$ is

(a) 2 ______________________________ (correct)

(b) ∞

(c) 4

(d) $\sqrt{2}$

(e) 0

$x^2 + 4 = 0 \Rightarrow x = \pm 2i$

$\text{dist}(0, \pm 2i) = |\pm 2i| = \sqrt{0 + 2^2} = \sqrt{4} = 2$

#9
§7.3

26. A 2×2 matrix A has an eigenvector $\begin{bmatrix} 1+2i \\ 2 \end{bmatrix}$ associated with the eigenvalue $\lambda = 4i$ of A. Then the general solution of the system $X' = AX$ is

(a) $X = c_1 \begin{bmatrix} \cos(4t) - 2\sin(4t) \\ 2\cos(4t) \end{bmatrix} + c_2 \begin{bmatrix} 2\cos(4t) + \sin(4t) \\ 2\sin(4t) \end{bmatrix}$ ____________(correct)

(b) $X = c_1 \begin{bmatrix} \cos(4t) + 3\sin(4t) \\ \cos(4t) \end{bmatrix} + c_2 \begin{bmatrix} \cos(4t) + \sin(4t) \\ \sin(4t) \end{bmatrix}$

(c) $X = c_1 \begin{bmatrix} -\cos(4t) + 2\sin(4t) \\ 2\cos(4t) \end{bmatrix} + c_2 \begin{bmatrix} 2\cos(4t) + \sin(4t) \\ 2\sin(4t) \end{bmatrix}$

(d) $X = c_1 \begin{bmatrix} 2\cos(4t) - \sin(4t) \\ 3\cos(4t) \end{bmatrix} + c_2 \begin{bmatrix} 2\cos(4t) - \sin(4t) \\ 3\sin(4t) \end{bmatrix}$

(e) $X = c_1 \begin{bmatrix} \cos(4t) + \sin(4t) \\ \cos(4t) \end{bmatrix} + c_2 \begin{bmatrix} \cos(4t) + 2\sin(4t) \\ -2\sin(4t) \end{bmatrix}$

$$X = \begin{bmatrix} 1+2i \\ 2 \end{bmatrix} e^{4it} = \begin{bmatrix} 1+2i \\ 2 \end{bmatrix} (\cos 4t + i \sin 4t)$$

$$= \begin{bmatrix} \cos 4t - 2\sin 4t + i(2\cos 4t + \sin 4t) \\ 2\cos 4t + i\, 2\sin 4t \end{bmatrix}$$

$$X_1 = Re(X) = \begin{bmatrix} \cos 4t - 2\sin 4t \\ 2\cos 4t \end{bmatrix}$$

$$X_2 = Im(X) = \begin{bmatrix} 2\cos 4t + \sin 4t \\ 2\sin 4t \end{bmatrix}$$

G.S. : $X = c_1 X_1 + c_2 X_2$

27. The matrix

#13
§7.6

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{bmatrix}$$

has a defective eigenvalue $\lambda = -1$ of defect 2. Choosing $v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ such that $(A+I)^3 v_3 = 0$ and $(A+I)^2 v_3 \neq 0$, then the general solution of the system $X' = AX$ is

(a) $X = \left(c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} t \\ 2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} t^2/2 \\ 1+2t \\ t \end{bmatrix} \right) e^{-t}$ ________________(correct)

(b) $X = \left(c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} t \\ 1 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} t^2/2 \\ 1+2t \\ t \end{bmatrix} \right) e^{-t}$

(c) $X = \left(c_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} t \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} t^2 \\ 2t \\ 2 \end{bmatrix} \right) e^{-t}$

(d) $X = \left(c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -t \\ 2 \\ -1 \end{bmatrix} t + c_3 \begin{bmatrix} t^2 \\ t \\ t \end{bmatrix} \right) e^{-t}$

(e) $X = \left(c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} t \\ 2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} t^2/2 \\ -t \\ 1+t \end{bmatrix} \right) e^{-t}$

$$V_2 = (A+I)V_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & -4 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$V_1 = (A+I)V_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & -4 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$X_1 = V_1 e^{-t} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{-t}$$

$$X_2 = (V_1 t + V_2) e^{-t} = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right) e^{-t} = \begin{bmatrix} t \\ 2 \\ 1 \end{bmatrix} e^{-t}$$

$$X_3 = \left(V_1 \frac{t^2}{2} + V_2 t + V_3\right) e^{-t} = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{t^2}{2} + \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) e^{-t} = \begin{bmatrix} t^2/2 \\ 2t+1 \\ t \end{bmatrix} e^{-t}$$

$$X = c_1 X_1 + c_2 X_2 + c_3 X_3$$

28. Let

#11
§ 8.1

$$A = \begin{bmatrix} 5 & -3 \\ 2 & 0 \end{bmatrix}, \quad \Phi(t) = \begin{bmatrix} e^{2t} & 3e^{3t} \\ e^{2t} & 2e^{3t} \end{bmatrix}$$

If $\Phi(t)$ is a fundamental matrix for the system $X' = AX$, then $e^{At} =$

(a) $\begin{bmatrix} -2e^{2t} + 3e^{3t} & 3e^{2t} - 3e^{3t} \\ -2e^{2t} + 2e^{3t} & 3e^{2t} - 2e^{3t} \end{bmatrix}$ ——— (correct)

(b) $\begin{bmatrix} 2e^{2t} + 3e^{3t} & 3e^{2t} + 3e^{3t} \\ 2e^{2t} + 2e^{3t} & 3e^{2t} + 2e^{3t} \end{bmatrix}$

(c) $\begin{bmatrix} 2e^{2t} - 3e^{3t} & -3e^{2t} + 3e^{3t} \\ 2e^{2t} - 2e^{3t} & -3e^{2t} + 2e^{3t} \end{bmatrix}$

(d) $\begin{bmatrix} -2e^{2t} + 9e^{3t} & e^{2t} - 3e^{3t} \\ -2e^{2t} + 6e^{3t} & e^{2t} - 2e^{3t} \end{bmatrix}$

(e) $\begin{bmatrix} e^{2t} + e^{3t} & e^{2t} - e^{3t} \\ e^{2t} - e^{3t} & e^{2t} + e^{3t} \end{bmatrix}$

- $\Phi(0) = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$
- $\Phi(0)^{-1} = \frac{1}{2-3}\begin{bmatrix} 2 & -3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}$
- $e^{At} = \Phi(t)\,\Phi(0)^{-1}$

$$= \begin{bmatrix} e^{2t} & 3e^{3t} \\ e^{2t} & 2e^{3t} \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -2e^{2t} + 3e^{3t} & 3e^{2t} - 3e^{3t} \\ -2e^{2t} + 2e^{3t} & 3e^{2t} - 2e^{3t} \end{bmatrix}$$

Answer KEY

Q	MASTER	VERSION01	VERSION02	VERSION03	VERSION04
1	A	D 7	E 5	E 3	A 5
2	A	A 9	A 8	B 4	A 6
3	A	E 2	A 7	C 2	B 9
4	A	C 8	A 3	C 5	A 4
5	A	C 4	E 1	D 7	B 1
6	A	B 3	E 9	C 10	D 3
7	A	B 10	C 10	E 1	B 8
8	A	D 6	E 6	A 6	B 10
9	A	B 1	A 2	B 8	E 2
10	A	E 5	B 4	E 9	B 7
11	A	C 14	D 20	E 19	A 11
12	A	B 15	E 19	B 18	B 13
13	A	D 16	C 13	E 15	D 12
14	A	B 12	A 12	C 12	B 19
15	A	E 20	B 16	E 13	B 14
16	A	A 11	A 17	B 11	C 18
17	A	A 18	B 11	B 14	E 20
18	A	D 17	E 15	A 20	E 17
19	A	D 19	C 14	C 17	D 16
20	A	D 13	A 18	B 16	D 15
21	A	C 22	E 22	B 22	B 23
22	A	E 24	E 21	D 24	C 21
23	A	E 23	A 24	B 23	D 22
24	A	E 21	E 23	B 21	D 24
25	A	C 26	B 26	C 25	A 28
26	A	A 28	E 25	B 26	A 25
27	A	E 25	E 27	E 27	C 26
28	A	B 27	C 28	A 28	B 27