1 [10pts]. (a) Two functions $f : \mathbb{Z}_7 \longrightarrow \mathbb{Z}_7$ and $g : \mathbb{Z}_7 \longrightarrow \mathbb{Z}_7$ are given by $f([a]) = [3a]$ and $g([a]) = [5a]$ for all $[a]$ in $\mathbb{Z}_7$.

(i) Find $f \circ g$ and $g \circ f$.

(ii) What can we deduce from (i)?

(b) Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be functions. Prove that if $g \circ f$ is a bijection, then g is onto.

2. [15pts] (a) For each of the following sets, indicate if it is finite, denumerable, or uncountable and justify your answers:

$\mathcal{P}(\mathbb{C})$, $\mathcal{P}(\mathbb{Z} \cap (0,7))$, $\mathbb{Z} \times (\mathbb{Q} \cap (0,7))$.

(b) Prove or disprove: If A and B are sets such that B is countable and $|A| < |B|$, then A is denumerable.

(d) Use Schröder-Bernstein Theorem to prove that $|\mathbb{R} - \mathbb{Z}| = |(0,1)|$.

3. [15pts] (a) Let G be the group $(\mathbb{Z}_8, +)$ and H be the group $(\mathbb{Z}_5^*, \cdot)$.

(i) Is $\{[0], [4]\}$ a subgroup of G?

(ii) Is $\{[0], [2], [4]\}$ a subgroup of H?

(b) Let G and G' be groups and $f : G \longrightarrow G'$ be an isomorphism. Prove that if G' is abelian, then G is also abelian.

(c) A group A has a subgroup B of order n. Suppose $4 \leq |A| \leq 14$ and that the number of distinct left cosets of B in A is $n + 2$. What is the value of n? Justify your answer.

4. [15pts] (a) Determine $\gcd(1197, 735)$ and find integers x, y such that

$$\gcd(1197, 735) = 1197x + 735y.$$

(b) Let $n \in \mathbb{N}$. Use the Fundamental Theorem of Arithmetic to prove that $n = 2^r a$, for some integers r and a where a is odd.

(c) Let $a \in \mathbb{N}$. Prove that $\gcd(a, 2a^2 + 1) = 1$.

(d) Let p be prime. Suppose there is an integer n such that

$$2n \equiv 1 \pmod{p} \qquad \text{and} \qquad n^2 \equiv -2 \pmod{p}.$$

Determine the value of p.

5. [15pts] (a) Let R be a relation on $\mathbb{R}$ defined by

$$aRb \text{ if and only if } a^2 - b^2 \text{ is an even integer.}$$

Prove that R is an equivalence relation.

(b) Let S be a relation on $\mathbb{N}$ defined by

$$aSb \text{ if and only if } a \text{ divides } b^2.$$

Is S reflexive? symmetric? antisymmetric? transitive? Justify your answers.

(c) Let T be a relation on $\mathbb{R}^+$ defined by

$$aTb \text{ if and only if } \frac{a^2}{b^2} \in \mathbb{N}.$$

Is $(\mathbb{R}^+, T)$ a poset? Is it well-ordered? Justify your answers.

[$\mathbb{R}^+$ is the set of positive real numbers.]