King Fahd University of Petroleum & Minerals
Department of Mathematics

Math 210 Introduction to Sets and Structures (Term 212)

FINAL EXAM
(Duration = 150 minutes | Number of Exercises = 15)

Exercise 1
Which of the following are partitions of $A = \{a, b, c, d, e, f, g\}$? For each collection of subsets that is not a partition of A, explain your answer.

(a) $S_1 = \{\{a, c, e, g\}, \{b, f\}, \{d\}\}$ (b) $S_2 = \{\{a, b, c, d\}, \{e, f\}\}$
(c) $S_3 = \{A\}$ (d) $S_4 = \{\{a\}, \emptyset, \{b, c, d\}, \{e, f, g\}\}$
(e) $S_5 = \{\{a, c, d\}, \{b, g\}, \{e\}, \{b, f\}\}$.

Exercise 2
In each of the following, two open sentences $P(x)$ and $Q(x)$ over a domain S are given. Determine all $x \in S$ for which $P(x) \Rightarrow Q(x)$ is a true statement.

(a) $P(x) : x - 3 = 4$; $Q(x) : x \geq 8$; $S = \mathbf{R}$.
(b) $P(x) : x^2 \geq 1$; $Q(x) : x \geq 1$; $S = \mathbf{R}$.
(c) $P(x) : x^2 \geq 1$; $Q(x) : x \geq 1$; $S = \mathbf{N}$.
(d) $P(x) : x \in [-1, 2]$; $Q(x) : x^2 \leq 2$; $S = [-1, 1]$.

Exercise 3
Let P and Q be statements. Show that $[(P \vee Q) \wedge \sim (P \wedge Q)] \equiv \sim (P \Leftrightarrow Q)$.

Exercise 4
Given below is a proof of a result. What is the result?

Proof Assume, without loss of generality, that x and y are even. Then $x = 2a$ and $y = 2b$ for integers a and b. Therefore,

$$xy + xz + yz = (2a)(2b) + (2a)z + (2b)z = 2(2ab + az + bz).$$

Since $2ab + az + bz$ is an integer, $xy + xz + yz$ is even. ■

Exercise 5
Let A and B be sets. Show, in general, that $\overline{A \times B} \neq \overline{A} \times \overline{B}$.

Exercise 6

(a) Prove that there exist two distinct primes p and q such that the four integers $pq \pm 2$ and $pq \pm 4$ are all primes.

(b) Disprove the statement: There exist two distinct primes p and q such that the six integers $pq \pm 2$, $pq \pm 4$ and $pq \pm 6$ are all primes.

Exercise 7

Consider the sequence $F_1, F_2, F_3, \ldots,$ where

$$F_1 = 1,\ F_2 = 1,\ F_3 = 2,\ F_4 = 3,\ F_5 = 5 \text{ and } F_6 = 8.$$

The terms of this sequence are called **Fibonacci numbers**.

(a) Define the sequence of Fibonacci numbers by means of a recurrence relation.

(b) Prove that $2 \mid F_n$ if and only if $3 \mid n$.

Exercise 8

Determine the maximum number of elements in a relation R on a 3-element set such that R has none of the properties reflexive, symmetric and transitive.

Exercise 9

The composition $g \circ f : (0, 1) \to \mathbf{R}$ of two functions f and g is given by $(g \circ f)(x) = \frac{4x-1}{2\sqrt{x-x^2}}$, where $f : (0, 1) \to (-1, 1)$ is defined by $f(x) = 2x - 1$ for $x \in (0, 1)$. Determine the function g.

Exercise 10

Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 4 & 1 & 5 & 3 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 6 & 2 & 1 & 4 \end{pmatrix}$ be elements of S_6.

(a) Determine α^{-1} and β^{-1}. (b) Determine $\alpha \circ \beta$ and $\beta \circ \alpha$.

Exercise 11

A function $f : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ is defined by $f(m, n) = 2^{m-1}(2n - 1)$.

(a) Prove that f is one-to-one and onto.

(b) Show that $\mathbf{N} \times \mathbf{N}$ is denumerable.

Exercise 12

True or False? Explain.

(a) If A is an uncountable set, then $|A| = |\mathbf{R}|$.

(b) There exists a bijective function $f : \mathbf{Q} \to \mathbf{R}$.

(c) If A, B and C are sets such that $A \subseteq B \subseteq C$ and A and C are denumerable, then B is denumerable.

(d) The set $S = \left\{ \frac{\sqrt{2}}{n} : n \in \mathbf{N} \right\}$ is denumerable.

(e) There exists a denumerable subset of the set of irrational numbers.

(f) Every infinite set is a subset of some denumerable set.

(g) If A and B are sets with the property that there exists an injective function $f : A \to B$, then $|A| = |B|$.

Exercise 13

(a) Prove for every pair p, q of distinct primes that $\sqrt{pq}$ is irrational.

(b) Prove for every pair p, q of distinct primes that $\sqrt{p} + \sqrt{q}$ is irrational.

Exercise 14

Let $(G, *)$ be a group with $G = \{a, b, c, d\}$, where a partially completed table for $(G, *)$ is given in Figure 15.6. Complete the table.

$*$	a	b	c	d
a	d	c		
b			a	
c				
d				

Exercise 15

Let S_3 denote the symmetric group of degree 3.

(a) Show that S_3 is NOT abelian.

(b) Consider the subgroup $H = \{\varepsilon, \sigma_1\}$ of S_3, where $\varepsilon = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ and $\sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$. Determine the distinct left cosets of H in S_3.