1. [12pts] (a) Let $a, b \in \mathbb{N}$ where a is even. Prove that either $a \nmid (b - 11)$ or $a \nmid (b + 22)$.

(b) (i) Let $r \in \mathbb{R}$. Prove that either $\sqrt{2} - r$ or $\sqrt{2} + r$ is irrational.

(ii) Prove that $\sqrt{2} + \sqrt{k}$ is irrational for every $k \in \mathbb{N}$.

2. [12pts] (a) Prove that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for each $n \in \mathbb{N}$

(b) A sequence $\{a_n\}_{n \in \mathbb{N}}$ is defined recursively by

$$a_1 = 1, \quad a_2 = 2, \quad a_n = 3a_{n-1} - 2a_{n-2} \ \text{ for } n \geq 3.$$

Make a conjecture about a_n and use strong induction to prove your conjecture.

3. [12pts]. (a) Let R be the relation on $\mathbb{Q} - \{0\}$ given by

$$aRb \quad \text{iff} \quad \frac{a - b}{ab} \in \mathbb{Z}.$$

Is R reflexive? symmetric? transitive? Justify your answers.

(b) Find an integer r such that $-8 \leq r \leq 8$ and $[19^{22}] = [r]$ in $\mathbb{Z}_{17}$.

4. [12pts] (a) Let b be a nonzero real number and let $f : \mathbb{R} - \{b\} \longrightarrow \mathbb{R}$ be the function given by $f(x) = \dfrac{x}{x - b}$. Is f one-to-one? Is it onto? Justify your answers.

(b) Let $g : \mathbb{Z}_9 \longrightarrow \mathbb{Z}_9$ be the function given by $g([x]) = [5x + 2]$ for each $[x] \in \mathbb{Z}_9$. Prove that g is a bijection.

(c) Let $h : \mathbb{Q} \longrightarrow \mathbb{Q}$ be a function such that $h \circ h = i_{\mathbb{Q}}$, the identity map on $\mathbb{Q}$. Prove that h is a bijection.