KFUPM/ Department of Mathematics/T221/MATH 210/ Final Exam/

Name: **ID #:** **Serial #:**

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1. [12pts] (a) Give an example of a denumerable subset X of $\mathbb{R}$ and an uncountable subset Y of $\mathbb{R}$ such that $X \cap Y$ is nonempty and finite.

(b) Let A and B be sets such that $A \subseteq B$. Prove that if A is uncountable, then B is uncountable.

(c) Using Schröder-Bernstein Theorem or otherwise, prove that $(0,1) \approx [0,1)$.

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2. [12pts] (a) Let p and q be distinct primes.

(i) Prove that $\gcd(p, p+q) = 1$.

(ii) Prove that if p and q are divisors of an integer a, then $pq|a$.

(b) State the fundamental theorem of arithmetic, and prove that if a and b are integers that are not coprime, then there exists a prime p dividing both a and b.

(c) Let m, n be integers such that $m \geq 2$, $2n \equiv 3 \pmod{m}$ and $2n^2 \equiv 7 \pmod{m}$. Find m.

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3. [12pts] (a) Let $\mathbb{Q}^*$ denote the multiplicative group of nonzero rational numbers and let G be the set of all rational numbers of the form 2^k, where $k \in \mathbb{Z}$. Prove that G is a subgroup of $\mathbb{Q}^*$.

(b) Let H be an Abelian group and let $f : H \longrightarrow H$ be the function given by $f(a) = a^{-1}$ for each a in H. Prove that f is an isomorphism.

(c) A group G of order 20 has a subgroup A of order α and a subgroup B of order β. If $10 \leq 2\alpha < \beta$, find all possible values of α and β.

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4. [12pts] (a) Let R be the relation on $\mathbb{Q}$ defined by aRb iff $3a - 3b \in \mathbb{Z}$. Prove that R is an equivalence relation and determine the equivalence class $[1/3]$.

(c) Let T be the relation on $\mathbb{R}^*$ defined by aTb iff $\dfrac{a}{b} = 2^k$ for some nonnegative integer k. Is $(\mathbb{R}^*, T)$ a poset? is it well-ordered? Justify your answers. [$\mathbb{R}^*$ is the set of all nonzero real numbers.]

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5. [12pts] Mark each of the following statements as True or False and justify your choices.

(a) For all positive integers a, b, c, if $a|bc$ then $a|b$ or $a|c$.

(b) The function $g : \mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{R}$ given by $g(x) = (x, 2x)$ for all $x \in \mathbb{R}$ is a bijection.

(c) The subset $\{[0], [3], [6], [9], [12], [15], [18]\}$ of $\mathbb{Z}_{20}$ is a cyclic subgroup of $\mathbb{Z}_{20}$ (under addition).