KFUPM/ Department of Mathematics/ T222/ MATH 210/ Exam 2/

Name: **ID #:** **Serial #:**

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1 [10pts] (a) Use contradiction to prove that $\sqrt[5]{2}$ is irrational.

Solution. Suppose for contradiction that $\sqrt[5]{2}$ is rational and let $\sqrt[5]{2} = \frac{a}{b}$, where $a, b \in \mathbb{N}$ and $\frac{a}{b}$ is in simplest form. Then $a^5 = 2b^5$, so that a^5 is even. This implies $a = 2k$, for some integer k and hence $2^5k^5 = 2b^5$, which gives $2^4k^5 = b^5$ and so b is also even, contradicting the fact that a and b are in simplest form. ■

(b) Let $a, b \in \mathbb{N}$, $a \geq 2$. Prove that either $a \nmid (b+1)$ or $a \nmid (2b+1)$.

Solution. Suppose for contradiction that $a|(b+1)$ and $a|(2b+1)$, then $a|(2(b+1) - (2b+1))$, i.e. $a|1$. This is impossible since $a \geq 2$ and the only divisors of 1 are 1 and -1. ■

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2 [10pts] (a) Use induction to prove that for each positive integer n

$$1 + 5 + 9 + \cdots + (4n-3) = 2n^2 - n$$

Solution. Let $P(n)$ be the statement to prove. Then $P(1)$ is true since the LHS and RHS of $P(1)$ are both equal to 1. Assume $P(n)$ is true for some $n \in \mathbb{N}$. We prove that $P(n+1)$ is true. We have

$$1 + 5 + \cdots + (4n-3) + (4(n+1)-3) = 2n^2 - n + 4n + 1 = 2n^2 + 3n + 1 = 2(n+1)^2 - (n+1). \blacksquare$$

(b) A sequence $\{a_n\}_{n\in\mathbb{N}}$ is defined recursively by

$$a_1 = 2,\ a_2 = 4,\ a_n = 5a_{n-1} - 6a_{n-2} \text{ for } n \geq 3.$$

Make a conjecture about a_n and use strong induction to prove your conjecture.

Solution. $a_1 = 2$, $a_2 = 4$, $a_3 = 5a_2 - 6a_1 = 8$. *Conjecture*: $\boxed{a_n = 2^n}$.
Proof. Conjecture is clearly true for $n = 1$ and $n = 2$. We use strong induction. Let $a_k = 2^k$, for all $k \in \{1, 2, \ldots, n\}$, where n is some integer greater than 2. We prove $a_{n+1} = 2^{n+1}$. We have

$$a_{n+1} = 5a_n - 6a_{n-1} = 5 \times 2^n - 6 \times 2^{n-1} = 5 \times 2^n - 3 \times 2^n = 2^{n+1}. \blacksquare$$

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3 [10pts] (a) Let R be the relation defined on $\mathbb{R}$ by

$$xRy \text{ if and only if } 7(x-y) \in \mathbb{Z}.$$

Is R reflexive? symmetric? transitive? Justify your answers.

Solution.

- $\forall x \in \mathbb{R}$, $7(x-x) = 0 \in \mathbb{Z}$, so $\boxed{R \text{ is reflexive}}$.
- $\forall x, y \in \mathbb{R}$, if $7(x-y) \in \mathbb{Z}$, then $7(y-x) \in \mathbb{Z}$, so $\boxed{R \text{ is symmetric}}$.
- $\forall x, y, z \in \mathbb{R}$, if $7(x-y) \in \mathbb{Z}$ and $7(y-z) \in \mathbb{Z}$, then $7(x-y) + 7(y-z) = 7(x-z) \in \mathbb{Z}$, so $\boxed{R \text{ is transitive}}$.

(b) Find the smallest nonnegative integers a, b, c such that

$$[-101] + [76] = [a], \quad [-101] \cdot [76] = [b], \quad \left[(-101)^{76}\right] = [c] \quad \text{in } \mathbb{Z}_7.$$

Solution. $-101 \equiv 4 \pmod 7$ and $76 \equiv -1 \pmod 7$, hence, in $\mathbb{Z}_7$

- $[-101] + [76] = [4] + [-1] = [3]$, so $\boxed{a = 3}$.
- $[-101] \cdot [76] = [4][-1] = [-4] = [3]$, so $\boxed{b = 3}$.
- $4^3 \equiv 1 \pmod 7$, so $4^{76} \equiv (4^3)^{25} \times 4 \equiv 4 \pmod 7$, so $\left[(-101)^{76}\right] = [4^{76}] = [4]$, i.e. $\boxed{c = 4}$.

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4 [10pts] (a) Let $a \in \mathbb{R}$ and let $f : \mathbb{R} - \{a\} \longrightarrow \mathbb{R} - \{a\}$ be the function given by

$$f(x) = a + \frac{1}{a - x}$$

Is f one-to-one? onto? Justify your answers.

Solution. f is one-to-one: Let $f(x) = f(x')$, where $x, x' \in \mathbb{R} - \{a\}$. Then $a + \dfrac{1}{a-x} = a + \dfrac{1}{a-x'}$, so that $a - x = a - x'$, i.e. $x = x'$.
f is onto: Let $y \in \mathbb{R} - \{a\}$, then $f\left(a + \dfrac{1}{a-y}\right) = a + \dfrac{1}{a - \left(a + \dfrac{1}{a-y}\right)} = y$, hence y is an image under f.
Another way: Let $y = f(x)$, then $y = a + \dfrac{1}{a-x}$, i.e. $x = a + \dfrac{1}{a-y} = f(y)$, which means $f \circ f$ is the identity map on $\mathbb{R} - \{a\}$ and so f is a bijection (equal to its own inverse). Hence $\boxed{f \text{ is one-to-one and onto}}$. ■

(b) Let A be a nonempty set and let f, g, h be functions from A to A such that $h \circ (g \circ f)$ is one-to-one. Prove that f is one-to-one.

Solution. Suppose $h \circ (g \circ f)$ is one-to-one, then $(h \circ g) \circ f$ is one-to-one. If $f(x) = f(y)$ for some $x, y \in A$, then $(h \circ g)(f(x)) = (h \circ g)(f(y))$, i.e. $((h \circ g) \circ f)(x) = ((h \circ g) \circ f)(y)$ and so $x = y$, as required. ■