King Fahd University of Petroleum & Minerals
Department of Mathematics
Math 225 Introduction to Linear Algebra
Final Exam - Term 221
(Duration = **2 h 30 min** | Number of Questions = **20 |** CODE 1)

Exercise 1 Given the linear systems

$$\text{(i)}\quad \begin{array}{lcl} x_1 + 2x_2 + x_3 & = & 2 \\ -x_1 - x_2 + 2x_3 & = & 3 \\ 2x_1 + 3x_2 & = & 1 \end{array} \quad \text{and} \quad \text{(ii)}\quad \begin{array}{lcl} x_1 + 2x_2 + x_3 & = & 0 \\ -x_1 - x_2 + 2x_3 & = & 2 \\ 2x_1 + 3x_2 & = & -2 \end{array}$$

If $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is the solution of (i) and $\begin{bmatrix} a' \\ b' \\ c' \end{bmatrix}$ is the solution of (ii), then $aa' + bb' + cc' =$

(**a**) 11 (**b**) -7 (**c**) 5 (**d**) -10 (**e**) -3

Exercise 2 Let A be a ***nonsingular*** 4×4 matrix. Then, the determinant of the adjoint of A is equal to

(**a**) $4\det(A)$ (**b**) $(\det(A))^4$ (**c**) $(\det(A))^2$ (**d**) $\det(A)$ (**e**) $(\det(A))^3$

Exercise 3 In the vector space $\mathbb{R}^{2\times 2}$, let A be a fixed matrix. Then:

(**a**) The set of all nonsingular matrices is NOT a subspace
(**b**) The set of all singular matrices is a subspace
(**c**) The set of all triangular matrices is a subspace
(**d**) The set of all symmetric matrices is NOT a subspace
(**e**) The set of all matrices that commute with A is NOT a subspace

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Exercise 4 In the vector space $C[0, 1]$, the Wronskian $W[x, \cos \pi x\, , \sin \pi x\,] =$

(**a**) π^3 (**b**) $\pi^3 x$ (**c**) x (**d**) 0 (**e**) $\pi^2 x$

Exercise 5 The polynomials $1+x+x^2$; $3+x+4x^2$; $a+bx^2$ form a basis for P_3 if and only if

(**a**) $2a-3b=0$ (**b**) $3a-2b\neq 0$ (**c**) $3a-2b=0$ (**d**) $3a+2b\neq 0$ (**e**) $2a-3b\neq 0$

Exercise 6 Consider the ordered bases $E=\{v_1,v_2,v_3\}$ and $F=\{u_1,u_2,u_3\}$ of $\mathbb{R}^3$, where

$$\boldsymbol{v_1}=\begin{bmatrix}1\\1\\0\end{bmatrix};\ \boldsymbol{v_2}=\begin{bmatrix}0\\1\\1\end{bmatrix};\ \boldsymbol{v_3}=\begin{bmatrix}1\\0\\1\end{bmatrix} \text{ and } \boldsymbol{u_1}=\begin{bmatrix}1\\0\\-1\end{bmatrix};\ \boldsymbol{u_2}=\begin{bmatrix}1\\-1\\1\end{bmatrix};\ \boldsymbol{u_3}=\begin{bmatrix}-1\\1\\0\end{bmatrix}.$$

Let $S=\begin{pmatrix}a & a' & a"\\ b & b' & b"\\ c & c' & c"\end{pmatrix}$ denote the transition matrix from E to F. Then, $aa'a"+bb'b"+cc'c"=$

(**a**) 18 (**b**) 28 (**c**) -14 (**d**) -9 (**e**) 35

Exercise 7 Let D be the differentiation operator on P_3 and consider the subspace $S=\{p\in P_3 \mid p(0)=0\}$. Then:

(**a**) $D\ :S\longrightarrow P_3$ is one-to-one
(**b**) $D:S\longrightarrow P_3$ is onto
(**c**) $D:\ P_3\longrightarrow P_2$ is one-to-one
(**d**) $D\ :P_3\longrightarrow P_2$ is NOT onto
(**e**) None of the above statements is true

Exercise 8 Let $E=\{u_1,u_2,u_3\}$ and $F=\{b_1,b_2\}$, where $u_1=\begin{bmatrix}1\\0\\-1\end{bmatrix}$; $u_2=\begin{bmatrix}1\\2\\1\end{bmatrix}$; $u_3=\begin{bmatrix}-1\\1\\1\end{bmatrix}$ and $b_1=\begin{bmatrix}1\\-1\end{bmatrix}$; $b_2=\begin{bmatrix}2\\-1\end{bmatrix}$.
Let L be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ defined by $L(\mathbf{x})=\begin{bmatrix}x_1+x_2\\x_1-x_3\end{bmatrix}$. The matrix representing L with respect to the ordered bases E and F is

(**a**) $\begin{pmatrix}-5 & 3 & 4\\ 3 & 3 & -2\end{pmatrix}$ (**b**) $\begin{pmatrix}5 & -3 & 4\\ 3 & 3 & -2\end{pmatrix}$ (**c**) $\begin{pmatrix}-5 & -3 & 4\\ 3 & 3 & 2\end{pmatrix}$ (**d**) $\begin{pmatrix}-5 & -3 & 4\\ 3 & 3 & -2\end{pmatrix}$ (**e**) $\begin{pmatrix}5 & -3 & 4\\ -3 & 3 & -2\end{pmatrix}$

Exercise 9 Let L be the linear operator on $\mathbb{R}^3$ defined $by\ L\ (x)\ =\ Ax$, where $A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & -2 & 1 \\ 1 & 1 & -1 \end{pmatrix}$ and let

$u_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$; $u_2 = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$; $u_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. The matrix representing L with respect to $\{u_1, u_2, u_3\}$ is

(**a**) $\begin{pmatrix} -1 & 8 & -6 \\ -1 & 3 & -4 \\ 2 & 7 & -4 \end{pmatrix}$ (**b**) $\begin{pmatrix} -1 & -8 & 6 \\ -1 & -3 & 4 \\ 2 & 7 & -4 \end{pmatrix}$ (**c**) $\begin{pmatrix} -1 & -8 & 6 \\ -1 & 3 & -4 \\ 2 & -7 & 4 \end{pmatrix}$ (**d**) $\begin{pmatrix} 1 & -8 & 6 \\ 1 & 3 & -4 \\ 2 & 7 & -4 \end{pmatrix}$ (**e**) $\begin{pmatrix} -1 & -8 & 6 \\ -1 & 3 & -4 \\ 2 & 7 & -4 \end{pmatrix}$

Exercise 10 Let $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ -1 \\ 0 \end{bmatrix}$. Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$ and $p = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ be the projection of $\mathbf{u}$ onto $\mathbf{v}$.

Then:

$a + b + c + d + \cos\theta =$

(**a**) $\dfrac{4+\sqrt{3}}{2}$ (**b**) $\dfrac{2+\sqrt{3}}{2}$ (**c**) $\dfrac{2+\sqrt{2}}{2}$ (**d**) $\dfrac{5}{2}$ (**e**) $\dfrac{3}{2}$

Exercise 11 Let u_1 and u_2 be an ***orthonormal*** basis for $\mathbb{R}^2$ and let u be a vector in $\mathbb{R}^2$ such that $\|u\| = 5$ and $|u^T u_1| = 3$, then $|u^T u_2| =$

(**a**) $\dfrac{\sqrt{2}}{3}$ (**b**) $\dfrac{\sqrt{3}}{2}$ (**c**) 2 (**d**) 4 (**e**) $\dfrac{3}{\sqrt{5}}$

Exercise 12 Let $A = \begin{pmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{pmatrix}$. An ***orthonormal*** basis for the column space of A is given by $\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$ and

(**a**) $\begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix}$ (**b**) $\dfrac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$ (**c**) $\dfrac{1}{2}\begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$ (**d**) $\dfrac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$ (**e**) $\begin{bmatrix} 2 \\ 2 \\ -2 \\ -2 \end{bmatrix}$

Exercise 13 Let $p_0, p_1, \ldots$ be a sequence of orthogonal polynomials and let α_n denote the lead coefficient of p_n. Then, $\|p_n\|^2 =$

(**a**) $|\alpha_n|\ \|x_n\|^2$ (**b**) $|\alpha_n|^2$ (**c**) $\alpha_n\langle p_n\ , x^n\rangle$ (**d**) 1 (**e**) $\dfrac{1}{|\alpha_n|^2}$

Exercise 14 Let $A = \begin{pmatrix} 3 & 1 & 2 \\ 0 & 1 & -2 \\ 0 & 1 & 4 \end{pmatrix}$. Then:

(**a**) A has three distinct eigenvalues and each has an eigenspace of dimension 1
(**b**) A has only two distinct eigenvalues λ_1 and λ_2 with dim(Eigenspace(λ_1)) $= 1$ and dim(Eigenspace(λ_2)) $= 2$
(**c**) A has only two distinct eigenvalues λ_1 and λ_2 with dim(Eigenspace(λ_1)) $= 2$ and dim(Eigenspace(λ_2)) $= 2$
(**d**) A has only one eigenvalue λ with multiplicity 3 and dim(Eigenspace(λ)) = 3
(**e**) A has only two distinct eigenvalues λ_1 and λ_2 with dim(Eigenspace(λ_1)) $= 1$ and dim(Eigenspace(λ_2)) $= 1$

Exercise 15 Let A be an 3×3 matrix with ***real*** entries. If A has a complex eigenvalue λ_1, then

(**a**) The eigenspace of λ_1 has dimension 2
(**b**) A has only two distinct eigenvalues
(**c**) A has no real eigenvalue
(**d**) A has three distinct eigenvalues
(**e**) λ_1 has multiplicity 2

Exercise 16 Let A and B be two $n \times n$ matrices and let λ be a nonzero eigenvalue of AB. Then:

(**a**) λ is an eigenvalue of B
(**b**)) $\frac{1}{\lambda}$ is an eigenvalue of A
(**c**) λ is an eigenvalue of $A^T B^T$
(**d**) $\frac{1}{\lambda}$ is an eigenvalue of $B^T A^T$
(**e**)) $\frac{1}{\lambda}$ is an eigenvalue of BA

Exercise 17 Let $A = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$. If we factor A into a product XDX^{-1}, where D is diagonal, then $X =$

(**a**) $\begin{pmatrix} 0 & 2 & 1 \\ 0 & -1 & -3 \\ 1 & 0 & 3 \end{pmatrix}$ (**b**) $\begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & 3 \end{pmatrix}$ (**c**) $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & -3 \\ 0 & -1 & 3 \end{pmatrix}$ (**d**) $\begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & -3 \\ 0 & 0 & 3 \end{pmatrix}$ (**e**) $\begin{pmatrix} 0 & 2 & 1 \\ 1 & -1 & -3 \\ 0 & 0 & 0 \end{pmatrix}$

Exercise 18 Let A be a $diagonalizable$ 6×6 matrix. If A has **only** two distinct eigenvalues λ and μ such that $\lambda I - A$ has rank 2, then the $multiplicity\ of\ \mu$ is equal to

(**a**) 5 (**b**) 1 (**c**) 3 (**d**) 4 (**e**) 2

Exercise 19 Let A be a $diagonalizable$ matrix whose eigenvalues are all either 1 or -1 and let $B = A + I$. Then, $B^3 =$

(**a**) $4B$ (**b**) $A^3 + 3I$ (**c**) B (**d**) $A^3 + A + I$ (**e**) I

Exercise 20 Consider the conic section $3x^2 - 2xy + 3y^2 + 8\sqrt{2}\,x - 2 = 0$. If a standard form for this quadratic equation is given by $ax'^2 + by'^2 = c$, then $(a, b, c) =$

(**a**) (4, 2, 8) (**b**) (2, 4, 2) (**c**) (4, 2, 14) (**d**) (2, 4, 16) (**e**) (2, 1, 8)