KFUPM/ Department of Mathematics/T222/MATH 225/Final Exam

Name: ID#: Serial #:

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1. [12pts] (a) Diagonalize, if possible, the matrix $A = \begin{bmatrix} 4 & -1 & 0 \\ 1 & 2 & 0 \\ -2 & 6 & 1 \end{bmatrix}$. If that is not possible, explain why.

(b) Find an orthogonal diagonalizing matrix Q for $B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

(c) Find (if any) the values of the real number k for which the matrix $C = \begin{bmatrix} 1 & 2 & 1 \\ 2 & k & 0 \\ 1 & 0 & k \end{bmatrix}$ is positive definite.

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2. [12pts] (a) Let U be the subspace of $\mathbb{R}^3$ (under the usual dot product) spanned by

$$\{(1,2,0),(0,1,1)\}.$$

Find a basis for the orthogonal complement $U^{\perp}$ of U.

(b) An inner product space V has orthonormal basis $\{w_1, w_2, w_3\}$. Let

$$a = 2w_1 + 2w_2 - 4w_3, \quad b = w_1 - 2w_2 + w_3, \quad c = xw_1 + yw_2 + zw_3$$

(i) Find the distance and the angle between a and b.

(ii) Find all possible values of x, y, z if $\|c\| = 3$, $\langle c, w_1\rangle = 1$, and $\langle c, w_2 + w_3\rangle = 0$.

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3. [12pts] (a) Use Gram-Schmidt process to transform the basis $\{(0,0,1),(1,2,0),(1,1,1)\}$ of $\mathbb{R}^3$ into an orthonormal basis (under the usual dot product).

(b) On the vector space $\mathbb{P}$ of all polynomials with real coefficients, consider the inner product given by

$$\langle p, q\rangle = \int_{-3}^{3} p(x)\,q(x)\,dx.$$

Determine b and c if $p_2(x) = x^2 + bx + c$ is orthogonal to $p_0(x) = 1$ and $p_1(x) = x$.

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4. [12pts] (a) For which values of the real number k does the equation

$$kx^2 + (k+3)y^2 + 4xy = 1$$

represent an ellipse? a hyperbola? Justify your answers.

(b) Determine a change of variables and use it to write in standard form the equation

$$3x^2 + 2yz = 1$$

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5. [12pts] Mark each of the following statements as **True** or **False** and justify your choices.

(a) If the RREF of an $n \times n$ matrix A has a zero row, then the system $Ax = \mathbf{0}$ has infinitely many solutions in $\mathbb{R}^n$.

(b) The function $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by $L(x,y) = (|x|, y)$ is a linear transformation.

(c) If A and B are similar matrices, then A^2 and B^2 are also similar.

(d) If a real matrix is diagonalizable, then its eigenvalues are distinct.

(e) If Q is an orthogonal matrix, then $Q = Q^{-1}$.

(f) If A is a positive definite symmetric matrix, then A is invertible.