Math 225 - 231 Second Major Exam Oct 26, 2023

Name: ______________________ ID #: ______________.

Q1) Let $\boldsymbol{V}$ be the set of ordered pairs (a, b) of real numbers with addition and scalar multiplication on $\boldsymbol{V}$ defined by

$$(a, b) + (c, d) = (a + c, b + d) \quad \text{and} \quad k(a, b) = (ka, 0)$$

Is $\boldsymbol{V}$ a vector space? Justify your answer.

Q2) Determine whether the subset $\boldsymbol{S}$ of $\boldsymbol{V}$ is a subspace of $\boldsymbol{V}$ if:

1. $\boldsymbol{V} = \boldsymbol{P_4}$ and $\boldsymbol{S}$ is the set of all polynomials in $\boldsymbol{P_4}$ having at least one real root.
2. $\boldsymbol{V} = \mathbb{R}^{2\times 2}$ and $\boldsymbol{S} = \left\{B \in \mathbb{R}^{2\times 2} \middle| AB \neq BA\right\}$ where A is a particular matrix in $\mathbb{R}^{2\times 2}$.

Q3) Let X_1, X_2 and X_3 be linearly independent vectors in $\mathbb{R}^n$ and let

$$Y_1 = X_1 + X_2, \quad Y_2 = X_2 + X_3 \quad \text{and} \quad Y_3 = X_3 + X_1$$

Are Y_1, Y_2 and Y_3 linearly independent? Justify your answer.

Q4) Consider the vector space $\mathbb{R}^{2\times 2}$, determine whether $B = \{A, B, C, D\}$ where

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}, \; B = \begin{pmatrix} 2 & 5 \\ 1 & -1 \end{pmatrix}, \; C = \begin{pmatrix} 5 & 12 \\ 1 & 1 \end{pmatrix} \text{ and } D = \begin{pmatrix} 3 & 4 \\ -2 & 5 \end{pmatrix}$$

form a basis for $\mathbb{R}^{2\times 2}$. Find the dimension of $\boldsymbol{span}(A, B, C, D)$.

Q5) Let $B_1 = \{1, 1 + x, 1 + x + x^2\}$ and $B_2 = \{1, 2x, 4x^2 - 2\}$ be two ordered bases of $\boldsymbol{P_3}$ and let v be a vector in $\boldsymbol{P_3}$ such that $[v]_{B_1} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$. Find $[v]_{B_2}$.

Q6) Show that the mapping $\mathrm{L}: \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$ defined by $L(A) = A - A^T$ is a linear operator on $\mathbb{R}^{\boldsymbol{3\times 3}}$. Find dim (ker(L)).