King Fahd University of Petroleum and Minerals
Department of Mathematics

MATH 225, Final Exam, Term 231

Duration: 150 minutes
02 January, 2024

Name:______________________________ ID Number:____________________________

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write legibly.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 9 pages of problems (Total of 9 Problems)

Question # Number	Points	Maximum Points
1		7
2		6
3		8
4		10
5		6
6		6
7		8
8		10
9		11
Total		72

1. [**7 points**] For what values of m and n, the system

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 2 \\ -x_1 - x_2 + 2x_3 &= 3 \\ 2x_1 + 3x_2 + mx_3 &= n \end{aligned}$$

has:
(i) a unique solution
(ii) no solution
(iii) infinitely many solutions.

2. [**6 points**] Use Crammer's rule to find the value of x_3 in the solution of the linear system

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 5 \\ 2x_1 + 2x_2 + x_3 &= 6 \\ x_1 + 2x_2 + 3x_3 &= 9 \end{aligned}$$

3. [**8 points**] (i) Determine whether the vectors $\vec{v}_1 = (4, 2, 3)^T$, $\vec{v}_2 = (2, 3, 1)^T$ and $\vec{v}_3 = (1, 1, -1)^T$ are linearly independent.

   (ii) Find a basis for the subspace $S = \text{Span}\,(\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3)$.

4. **[10 points]** Consider the mapping $L : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $L(x_1, x_2, x_3)^T = (x_1 - x_2 + x_3,\ x_2 - x_3 + x_1)^T$.

(i) Show that L is a linear transformation.
(ii) Find a basis for ker (L).
(iii) Find the matrix that represents L with respect to the ordered bases $E = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ and $F = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}$ of $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively.
(iv) Find the coordinate vector $\left[L \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right]_F$.

5. [**6 points**] Find the projection of the vector $f(x) = x^2$ on the subspace of P_3 spaned by the two vectors $g(x) = 1$ and $h(x) = x$ if the inner product on P_3 is defined by $< p, q >= \sum_{i=1}^{3} p(x_i)q(x_i)$, where $x_1 = -1$, $x_2 = 0$ and $x_3 = 1$.

6. [**6 points**] Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 6 \end{bmatrix}$.

Find an orthonormal basis for the column space of A.

7. [**8 points**] Let A be a 2×2 matrix.

(i) If tr $(A) = 8$ and $|A| = 12$, find the eigenvalues of A.

(ii) If $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are eigenvectors belonging to the eigenvalue λ_1 and λ_2 respectively, where $\lambda_1 \geq \lambda_2$, then find the matrix A.

8. [**10 points**] Let $A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 3 \\ 1 & 1 & -1 \end{bmatrix}$

(i) Find all the eigenvalues of A and the corresponding eigenvectors.

(ii) Is A diagonalizable? (Justify your answer)

(iii) Find the eigenvalues of A^{10} and the corresponding eigenvectors.

9. **[11 points]** For the quadratic equation $-3x^2 + 6xy + 5y^2 - 24 = 0$ find:

(i) The matrix form for this equation.

(ii) A suitable change of coordinates so that the resulting equation represents a conic section in standard form.

(iii) Identify the graph

(iv) Find the coordinates of the vertices of the graph.