**King Fahd University of Petroleum and Minerals**

**Department of Mathematics**

**Math 225, Final Exam, Term 241**

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1. **[7 points]** Let $\mathbb{x}, \mathbb{y} \in \mathbb{R}^4$. Show that if $A = \mathbb{x}\mathbb{y}^T$, then $\det(A) = 0$.
2. **[7 points]** Determine whether the following set is a basis for $M_{2\times2}$:

$$\left\{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}-1 & 1\\-1 & 1\end{bmatrix}, \begin{bmatrix}-1 & -1\\1 & 1\end{bmatrix}\right\}.$$

3. **[7 points]** Let $v_1, v_2, v_3$ be linearly independent vectors in $\mathbb{R}^3$. Find, if possible, a linear operator $L: \mathbb{R}^3 \to \mathbb{R}^3$ such that $L(v_1), L(v_2), L(v_3)$ are linearly dependent in $\mathbb{R}^3$.
4. **[7 points]** Find a basis for $S^\perp$ if $S$ is the subspace of $\mathbb{R}^4$ spanned by

$$X = \begin{bmatrix}1\\0\\2\\1\end{bmatrix}, Y = \begin{bmatrix}-1\\1\\0\\3\end{bmatrix}.$$

   Use your answer to find a basis for $\mathbb{R}^4$.

5. **[6 points]** Let $\mathbb{P}_3$ be the vector space of all polynomials of degree less than 3 equipped with the inner product

$$\langle p, q\rangle = \int_0^1 p(x)\, q(x)\, dx,\ p, q\ \in \mathbb{P}_3.$$

   a. Find the angle between $p(x) = 2\ and\ q(x) = x$.
   b. Find the projection of $f(x) = 2 - 3x$ onto $q(x) = x$.
6. **[5 points]** Let $Q$ be an $n \times n$ orthogonal matrix. Show that if $\{v_1, v_2, \cdots, v_n\}$ is an orthonormal basis for $\mathbb{R}^n$, then so is$\{Qv_1, Qv_2, \cdots, Qv_n\}$.
7. **[7 points]** Consider the vector space $\mathbb{R}^3$ equipped with the standard inner product. Find the vector projection of $v$ onto the subspace $U = Span(u_1, u_2)$, where

$$v = \begin{bmatrix}-2\\1\\1\end{bmatrix}, u_1 = \begin{bmatrix}3\\2\\6\end{bmatrix}, u_2 = \begin{bmatrix}4\\0\\-2\end{bmatrix}.$$

8. **[10 points]** Find a $QR$ factorization of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

Hint: First apply the Gram-Schmidt orthogonalization process on the columns of $A$.

9. **[7 points]** Find the eigenvalues and the corresponding eigenspaces for the matrix

$$A = \begin{bmatrix} -2 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}.$$

10. **[10 points]** Find $A^n$ if $A = \begin{bmatrix} 5 & -4 \\ 3 & -2 \end{bmatrix}$, where $n$ is a positive integer.

11. **[7 points]** Suppose $\mathbb{x}$ is an eigenvector a matrix $A$ corresponding to an eigenvalue $\lambda$. Show that $\mathbb{x}$ is an eigenvector of the matrix $A^3 - 2A + 3I$. What is the corresponding eigenvalue?

12. **[10 points]** Identify the graph of the following quadratic equation by first making a suitable substitution to remove the $xy - term$:

$$x^2 + 4xy + y^2 + x = 0.$$

13. **[10 points] True or False?**

    a. Let $A = [\mathbb{a}_1 \quad \mathbb{a}_2 \quad \mathbb{a}_3]$ and $b = 3\mathbb{a}_2 - 2\mathbb{a}_3$. Then the system $AX = b$ is consistent.

    b. If $A$ is a $7 \times 5$ matrix of rank 4, then the dimension of the null space of $A$ is 3.

    c. Let $V$ be a vector space and let $v_1, v_2, v_3 \in V$. If $v_1$ *and* $v_2$ are L. independent and $v_1$ *and* $v_3$ are L. independent, then $v_1, v_2, v_3$ are L. independent.

    d. Let $A$ be an $m \times n$ matrix. Then $Col(A) = N(A^T)^{\perp}$.

    e. Each eigenvector of a square matrix $A$ is also an eigenvector of $A^2$.

Good luck,

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