King Fahd University of Petroleum and Minerals
Department of Mathematics

Math 225 (Term 251)

Exam 1 (90 minutes - 12 Exercises)

CODE 1

Exercise 1. Given the linear systems

$$\text{(i)} \quad \begin{aligned} x_1 + 2x_2 + x_3 &= 2 \\ -x_1 - x_2 + 2x_3 &= 3 \\ 2x_1 + 3x_2 &= 0 \end{aligned} \qquad \text{and} \qquad \text{(ii)} \quad \begin{aligned} x_1 + 2x_2 + x_3 &= -1 \\ -x_1 - x_2 + 2x_3 &= 2 \\ 2x_1 + 3x_2 &= -2 \end{aligned}$$

If $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is the solution of (i) and $\begin{bmatrix} a' \\ b' \\ c' \end{bmatrix}$ is the solution of (ii), then $aa' + bb' + cc' =$

(a) -10

(b) -7

(c) -9

(d) 5

(e) 11

Exercise 2. Let A be an $n \times n$ matrix such that $A^k = 0$, where $k \geq 2$. Then, $I - A$ is nonsingular with $(I - A)^{-1} =$

(a) $I - A^{-1}$

(b) $I + A^{k-1}$

(c) $I + A^{k-2}$

(d) $I + A + A^2 + \cdots + A^{k-1}$

(e) $I + A + A^2 + \cdots + A^{k-2}$

Exercise 3. Let $A = \begin{pmatrix} -2 & 1 & 2 \\ 4 & 1 & -2 \\ -6 & -3 & 4 \end{pmatrix}$ and assume the LU factorization of the matrix A is given by: $L = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a & a_{22} & a_{23} \\ b & c & a_{33} \end{pmatrix}$ *and* $U = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & d \end{pmatrix}$.

Then, $(a\ ,\ b\ ,\ c\ ,\ d)\ =$

(a) $(-2\ ,\ 3\ ,\ -2\ ,\ -2)$

(b) $(-2\ ,\ 3\ ,\ -2\ ,\ 2)$

(c) $(-2\ ,\ 3\ ,\ 2\ ,\ 2)$

(d) $(2\ ,\ 3\ ,\ 2\ ,\ 2)$

(e) $(2\ ,\ -3\ ,\ 2\ ,\ 2)$

Exercise 4. Let $A = \begin{pmatrix} a-x & b & c \\ 1 & -x & 0 \\ 0 & 1 & -x \end{pmatrix}$. Then, $\det(A)\ =$

(a) $x^3 + ax^2 + bx + c$

(b) $x^3 + ax^2 + bx - c$

(c) $-x^3 - ax^2 + bx + c$

(d) $-x^3 + ax^2 - bx + c$

(e) $-x^3 + ax^2 + bx + c$

Exercise 5. Recall that a set $(V, +, \cdot)$ is called a vector space over $\mathbb{R}$ if V is closed under addition and scalar multiplication and the following axioms are satisfied:

A1 $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$ for any $\mathbf{x}$ and $\mathbf{y}$ in V.
A2 $(\mathbf{x}+\mathbf{y})+\mathbf{z} = \mathbf{x}+(\mathbf{y}+\mathbf{z})$ for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$.
A3 There exists an element $\mathbf{0} \in V$ such that $\mathbf{x}+\mathbf{0} = \mathbf{x}$ for each $\mathbf{x} \in V$.
A4 For each $\mathbf{x} \in V$, there exists an element $-\mathbf{x} \in V$ such that $\mathbf{x}+(-\mathbf{x}) = \mathbf{0}$.
A5 $\alpha(\mathbf{x}+\mathbf{y}) = \alpha\mathbf{x}+\alpha\mathbf{y}$ for each scalar $\alpha \in \mathbb{R}$ and any $\mathbf{x}, \mathbf{y} \in V$.
A6 $(\alpha+\beta)\mathbf{x} = \alpha\mathbf{x}+\beta\mathbf{x}$ for any scalars $\alpha, \beta \in \mathbb{R}$ and any $\mathbf{x} \in V$.
A7 $(\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$ for any scalars $\alpha, \beta \in \mathbb{R}$ and any $\mathbf{x} \in V$.
A8 $1 \cdot \mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in V$.

Let $\mathbb{Z}$ denote the set of all integers with addition defined in the usual way, and define scalar multiplication, denoted $\circ$, by

$$\alpha \circ k = [\![\alpha]\!] \cdot k \quad \text{for all } k \in \mathbb{Z},$$

where $[\![\alpha]\!]$ denotes the greatest integer less than or equal to α. For example,

$$2.25 \circ 4 = [\![2.25]\!] \cdot 4 = 2 \cdot 4 = 8.$$

$\big(\mathbb{Z}\ ,\ +\ ,\ \circ\big)$ is Not a vector space. Which axioms **fail** to hold?

(a) A2 and A6

(b) A2 and A7

(c) A5 and A6

(d) A5 and A7

(e) A6 and A7

Exercise 6. In the vector space $C[0,1]$, the Wronskian $W\big[x, \cos(\pi x), \sin(\pi x)\big] =$

(a) 0

(b) x

(c) π^3

(d) $\pi^2 x$

(e) $\pi^3 x$

Exercise 7. Let $A = \begin{pmatrix} 2 & 3 & 6 \\ 1 & 2 & -1 \\ -2 & -3 & -6 \end{pmatrix}$. Then, the null space of A is given by $N(A) =$

(a) $\text{Span}\left(\begin{pmatrix} -30 \\ 16 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}\right)$

(b) $\text{Span}\left(\begin{pmatrix} 15 \\ -8 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ -3 \end{pmatrix}\right)$

(c) 0

(d) $\text{Span}\left(\begin{pmatrix} -15 \\ 8 \\ 1 \end{pmatrix}\right)$

(e) $\text{Span}\left(\begin{pmatrix} 5 \\ 8 \\ 0 \end{pmatrix}\right)$

Exercise 8. Given the vectors

$$\mathbf{x}_1 = \begin{pmatrix} -4 \\ 6 \\ -2 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 2 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 6 \\ -9 \\ 3 \\ 0 \end{pmatrix}, \quad \mathbf{x}_4 = \begin{pmatrix} -2 \\ 3 \\ -1 \\ 0 \end{pmatrix}$$

The dimension of $\text{Span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)$ is equal to:

(a) 0

(b) 1

(c) 2

(d) 3

(e) 4

Exercise 9. Let P_3 denote the space of all real polynomials of degree ≤ 2. Let S be the subspace of P_3 consisting of all polynomials $p(x)$ such that $p(0) = 0$, and let T be the subspace of P_3 consisting of all polynomials $q(x)$ such that $q(1) = 0$. If $\dim(S) = s$, $\dim(T) = t$, and $\dim(S \cap T) = r$, then $(s\ ,\ t\ ,\ r)$ =

(a) (1 , 2 , 1)
(b) (2 , 1 , 1)
(c) (2 , 2 , 0)
(d) (2 , 2 , 1)
(e) (2 , 3 , 1)

Exercise 10. Let

$$\mathbf{v}_1 = \begin{pmatrix} 4 \\ 6 \\ 7 \end{pmatrix},\ \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},\ \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}.$$

The transition matrix from the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to the basis $F = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is

(a) $\begin{pmatrix} 1 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & -1 & 2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & -1 & -2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & -1 & -2 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & -1 & -2 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$

Exercise 11. Let

$$A = \begin{pmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 13 & 5 \end{pmatrix}$$

Find the dimension of the column space of A, and identify a basis for it.

(a) Dimension = 2, Basis = Columns 1 and 2

(b) Dimension = 3, Basis = Columns 1, 2, and 5

(c) Dimension = 4, Basis = Columns 1, 2, 3, and 5

(d) Dimension = 3, Basis = Columns 1, 3, and 4

(e) Dimension = 3, Basis = Columns 2, 3, and 4

Exercise 12. Let

$$A = \begin{pmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{pmatrix}.$$

Then:

(a) $\text{rank}(A) = 1$, $\text{nullity}(A) = 3$

(b) $\text{rank}(A) = 2$, $\text{nullity}(A) = 1$

(c) $\text{rank}(A) = 2$, $\text{nullity}(A) = 2$

(d) $\text{rank}(A) = 3$, $\text{nullity}(A) = 0$

(e) $\text{rank}(A) = 3$, $\text{nullity}(A) = 1$