King Fahd University of Petroleum and Minerals
Department of Mathematics

Math 225 (Term 251)

Exam 2 (90 minutes - 12 Exercises)

CODE 1

Exercise 1. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator defined by

$$L(\mathbf{x}) = \begin{pmatrix} x_1 - x_3 \\ x_2 - x_3 \\ x_1 - x_2 \end{pmatrix}$$

Then, the kernel and range of L are given by

(a) $\ker(L) = \mathrm{Span}(e_1, e_2, e_3)$, $\mathrm{range}(L) = \mathrm{Span}(e_1 + e_3, e_2 - e_3)$

(b) $\ker(L) = \mathrm{Span}(e_1 + e_2 + e_3)$, $\mathrm{range}(L) = \mathrm{Span}(e_1 - e_3, e_2 - e_3)$

(c) $\ker(L) = \mathrm{Span}(e_1, e_2, e_3)$, $\mathrm{range}(L) = \mathrm{Span}(e_1 + e_3, e_2 + e_3)$

(d) $\ker(L) = \mathrm{Span}(e_1 + e_2 + e_3)$, $\mathrm{range}(L) = \mathrm{Span}(e_1 + e_3, e_2 - e_3)$

(e) $\ker(L) = \mathrm{Span}(e_1, e_2)$, $\mathrm{range}(L) = \mathrm{Span}(e_1 - e_3, e_2 - e_3)$

Exercise 2. Let P_n denote the space of all real polynomials of degree $\leq n-1$, for any integer $n \geq 1$. Let D be the differentiation operator on P_3, and consider the subspace $S = \{p \in P_3 \mid p(0) = 0\}$. Then:

(a) $D : S \to P_3$ is onto

(b) $D : S \to P_3$ is one-to-one

(c) $D : P_3 \to P_2$ is one-to-one

(d) $D : P_3 \to P_2$ is NOT onto

(e) None of the above statements is true

Exercise 3. Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by $L(\mathbf{x}) = \begin{pmatrix} x_1 + x_2 \\ x_2 - x_3 \end{pmatrix}$ and let $E = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $F = \{\mathbf{b}_1, \mathbf{b}_2\}$, where

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

The matrix representing L with respect to the ordered bases E and F is given by

(a) $\begin{pmatrix} -3 & 2 \\ -5 & 4 \\ 0 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 3 & 0 \\ 2 & 0 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 0 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 2 & 0 \\ -1 & -1 & 0 \end{pmatrix}$ (e) $\begin{pmatrix} -3 & -5 & 0 \\ 2 & 4 & 0 \end{pmatrix}$

Exercise 4. Let L be the operator on P_3 defined by $L(p(x)) = xp'(x) + p''(x)$. The matrix representing L with respect to the ordered basis $\{1,\ x,\ x + x^2\}$ is given by

(a) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (e) $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Exercise 5. Let $\mathbf{x}$ and $\mathbf{y}$ be *linearly independent* vectors in $\mathbb{R}^2$. Suppose that $\|\mathbf{x}\| = 2$ and $\|\mathbf{y}\| = \frac{1}{2}$. Which of the following is a *possible* value for $\mathbf{x}^T\mathbf{y}$?

(a) 1 (b) $\frac{3}{2}$ (c) $\sqrt{\frac{3}{4}}$ (d) $\frac{5}{2}$ (e) $\sqrt{\frac{17}{4}}$

Exercise 6. Let $A = \begin{pmatrix} -1 & 0 & 2 \\ 3 & 0 & -6 \end{pmatrix}$ and let $R(A)$ denote the column space of A. Then, the *orthogonal complement* of $R(A)$ is spanned by

(a) $\begin{pmatrix} 2 \\ 6 \end{pmatrix}$ (b) $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ (c) $\begin{pmatrix} 6 \\ -2 \end{pmatrix}$ (d) $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ (e) $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Exercise 7. Given n distinct real numbers $x_1, \ldots, x_n$, for each pair of polynomials $p, q \in P_n$, define the inner product by

$$\langle p, q \rangle = \sum_{i=1}^{n} p(x_i) q(x_i).$$

In P_5, let $x_i = \dfrac{i-1}{2}$ for $i = 1,2,3,4,5$. Then, the *length* (or *norm*) of the polynomial $p(x) = 4x$ is equal to

(a) $\sqrt{30}$ (b) $\sqrt{31}$ (c) $\sqrt{32}$ (d) $\sqrt{120}$ (e) $\sqrt{124}$

Exercise 8. Let $x \in \left(0, \dfrac{\pi}{2}\right)$ and let A be an $m \times m$ matrix of the form

$$A = \begin{pmatrix} 1 & -\cos x & -\cos x & \cdots & -\cos x & -\cos x \\ 0 & \sin x & -\sin x \cos x & \cdots & -\sin x \cos x & -\sin x \cos x \\ 0 & 0 & \sin^2 x & \cdots & -\sin^2 x \cos x & -\sin^2 x \cos x \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \sin^{m-2} x & -\sin^{m-2} x \cos x \\ 0 & 0 & 0 & \cdots & 0 & \sin^{m-1} x \end{pmatrix}$$

The Frobenius norm $\|A\|_F$ of this matrix is equal to

(a) 1 (b) $\sqrt{m-1}$ (c) $m-1$ (d) $\sqrt{m}$ (e) m

Exercise 9. Let $\mathbf{u}_1 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} \end{pmatrix}$ and $\mathbf{u}_2 = \begin{pmatrix} \sqrt{\frac{2}{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix}$ be a basis of $\mathbb{R}^2$. Write the vector $\mathbf{x} = \begin{pmatrix} \sqrt{3} \\ \sqrt{6} \end{pmatrix}$ in the basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ as $\mathbf{x} = a\mathbf{u}_1 + b\mathbf{u}_2$. Then, $a + b =$

(a) 2 (b) 3 (c) $3 + 2\sqrt{2}$ (d) $2\sqrt{2}$ (e) $2 + 2\sqrt{3}$

Exercise 10. Let $A = \begin{pmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{pmatrix}$.

An *orthonormal* basis for the column space of A is given by

(a) $\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$ (b) $\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$ (c) $\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \frac{5}{2}\begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, 2\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$

(d) $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \frac{5}{2}\begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, 2\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$ (e) $\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$

Exercise 11. In $C[-\pi,\pi]$, consider the inner product defined by

$$\langle f,g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx.$$

Use an orthonormal set in $C[-\pi,\pi]$ to evaluate $\displaystyle\int_{-\pi}^{\pi}\cos^4(x)\,dx\ =$

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{4}$ (d) π (e) $\frac{5\pi}{4}$

Exercise 12. **Theorem** Let $p_0, p_1, \ldots$ be a sequence of orthogonal polynomials. Let a_i denote the lead coefficient of p_i for each i, and define $p_{-1}(x)$ to be the zero polynomial. Then

$$\alpha_{n+1}p_{n+1}(x) = (x-\beta_n)p_n(x) - \alpha_n\gamma_n p_{n-1}(x) \qquad (n \geq 0)$$

where

$$\alpha_0 = \gamma_0 = 1\,, \quad \alpha_n = \frac{a_{n-1}}{a_n}\,, \quad \beta_n = \frac{\langle p_{n-1}, xp_{n-1}\rangle}{\langle p_{n-1}, p_{n-1}\rangle}\,, \quad \gamma_n = \frac{\langle p_n, p_n\rangle}{\langle p_{n-1}, p_{n-1}\rangle} \quad (n \geq 1). \ \blacksquare$$

Now, let p_0, p_1, p_2 be orthogonal polynomials with respect to the inner product

$$\langle p(x), q(x)\rangle = \int_{-1}^{1}\frac{p(x)q(x)}{1+x^2}dx.$$

Assume all polynomials have lead coefficient -1. If we use the above theorem to calculate p_1 and p_2, then a correct choice for $\big(p_1, p_2\big)$ is:

(a) $\big(p_1,p_2\big) = \big(-x-1,\ -x^2+1\big)$

(b) $\big(p_1,p_2\big) = \Big(-x+1,\ -x^2-\frac{2}{\pi}\Big)$

(c) $\big(p_1,p_2\big) = \Big(-x,\ -x^2+\frac{2}{\pi}-1\Big)$

(d) $\big(p_1,p_2\big) = \Big(-x,\ -x^2+\frac{4}{\pi}-1\Big)$

(e) $\big(p_1,p_2\big) = \Big(-x,\ -x^2-\frac{4}{\pi}+1\Big)$