

King Fahd University of Petroleum and Minerals
Department of Mathematics

Math 225 (Term 251)

Final (150 minutes - 20 Exercises)

Code 1

Exercise 1. Let $A = \begin{pmatrix} -1 & 0 & 3 \\ 0 & 2 & 0 \\ 1 & -2 & 3 \end{pmatrix}$ and assume $A^{-1} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.

Then, $(a_{11}, a_{22}, a_{33}) =$

- (a) $\left(-\frac{1}{2}, \frac{1}{6}, \frac{1}{6}\right)$ (b) $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{6}\right)$ (c) $\left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{6}\right)$
(d) $\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{6}\right)$ (e) $\left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{6}\right)$

Exercise 2. Let

$$A = \begin{pmatrix} 2 & 1 & 2 & 2 \\ -3 & 3 & 1 & 3 \\ 2 & 1 & -1 & -4 \\ 1 & -3 & 2 & 0 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & -3 & 2 & 0 \\ 2 & 1 & -1 & -4 \\ 2 & 1 & 2 & 2 \\ -3 & 3 & 1 & 3 \end{pmatrix}; \quad C = \begin{pmatrix} 2 & 1 & 2 & 2 \\ -3 & 3 & 1 & 3 \\ -1 & 4 & 0 & -1 \\ -2 & 0 & 3 & 3 \end{pmatrix}$$

By analyzing the effect of the row operations applied to A on its determinant, we find that $\det(B) - \det(C) =$

- (a) $2\det(A)$ (b) $\det(A)$ (c) 0 (d) $-\det(A)$ (e) $-2\det(A)$

Exercise 3. Let $\mathbb{R}^{2 \times 2}$ denote the set of all 2×2 matrices with real entries, P_4 denote the set of all real polynomials of degree ≤ 3 , and $C^2[a, b]$ denote the set of all functions that have a continuous second derivative on the interval $[a, b]$.

Which one of the following assertions about subspaces is correct?

- (a) The set of all singular 2×2 matrices is a subspace of $\mathbb{R}^{2 \times 2}$.
- (b) The set of all polynomials in P_4 having at least one real root is a subspace of P_4 .
- (c) The set of all solutions on $[a, b]$ to the differential equation $y'' + y = 0$ is a subspace of $C^2[a, b]$.
- (d) The set of all polynomials in P_4 of degree exactly 3 is a subspace of P_4 .
- (e) The set of all triangular 2×2 matrices is a subspace of $\mathbb{R}^{2 \times 2}$.

Exercise 4. In the vector space $C[-\pi, \pi]$, the Wronskian $W\left[1, \sin(x), \cos^2\left(\frac{x}{2}\right)\right] =$

- (a) -1 (b) $-\frac{1}{2}$ (c) 0 (d) $\frac{1}{2}$ (e) 1
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Exercise 5. Given

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad S = \begin{pmatrix} 3 & 4 \\ 3 & 5 \end{pmatrix}$$

If S is the transition matrix from the basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ to another basis $\{\mathbf{u}_1, \mathbf{u}_2\}$, then

(a) $\mathbf{u}_1 = \begin{pmatrix} 12 \\ 21 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 18 \\ 31 \end{pmatrix}$

(b) $\mathbf{u}_1 = \frac{1}{3} \begin{pmatrix} 4 \\ 11 \end{pmatrix}, \quad \mathbf{u}_2 = \frac{1}{3} \begin{pmatrix} -2 \\ -7 \end{pmatrix}$

(c) $\mathbf{u}_1 = \frac{1}{3} \begin{pmatrix} 2 \\ 8 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

(d) $\mathbf{u}_1 = \frac{1}{3} \begin{pmatrix} 5 \\ 3 \end{pmatrix}, \quad \mathbf{u}_2 = \frac{1}{3} \begin{pmatrix} -4 \\ 6 \end{pmatrix}$

(e) $\mathbf{u}_1 = \frac{1}{3} \begin{pmatrix} 2 \\ 7 \end{pmatrix}, \quad \mathbf{u}_2 = \frac{1}{3} \begin{pmatrix} -5 \\ -1 \end{pmatrix}$

Exercise 6. Let A be a 4×5 matrix with reduced row echelon form

$$U = \begin{pmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 3 & 0 & -2 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and suppose that the first, second, and fourth columns of A are:

$$\mathbf{a}_1 = \begin{pmatrix} 2 \\ 1 \\ -3 \\ -2 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} -1 \\ 2 \\ 3 \\ 1 \end{pmatrix}, \quad \mathbf{a}_4 = \begin{pmatrix} -2 \\ -1 \\ 3 \\ 4 \end{pmatrix}.$$

Then, the third and fifth columns of A are given by

(a) $\begin{pmatrix} 1 \\ -4 \\ -15 \\ -7 \end{pmatrix}, \begin{pmatrix} 8 \\ 7 \\ 25 \\ 24 \end{pmatrix}$ (b) $\begin{pmatrix} -1 \\ 7 \\ 6 \\ 1 \end{pmatrix}, \begin{pmatrix} -9 \\ -8 \\ 14 \\ 22 \end{pmatrix}$ (c) $\begin{pmatrix} 7 \\ 8 \\ -3 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 11 \\ 29 \\ 11 \end{pmatrix}$ (d) $\begin{pmatrix} 1 \\ 8 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} -10 \\ -10 \\ 12 \\ 20 \end{pmatrix}$ (e) $\begin{pmatrix} 4 \\ 7 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -12 \\ -11 \\ 17 \\ 18 \end{pmatrix}$

Exercise 7. Let $L : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be the linear operator defined by

$$L(\mathbf{x}) = \begin{pmatrix} x_3 \\ x_1 \\ x_3 \\ x_2 \end{pmatrix}$$

and let S be the subspace of \mathbb{R}^4 spanned by \mathbf{e}_3 and \mathbf{e}_4 . Then:

- (a) $\dim(L(S)) = 0$ (b) $\dim(L(S)) = 1$ (c) $\dim(L(S)) = 2$
(d) $\dim(L(S)) = 3$ (e) $\dim(L(S)) = 4$

Exercise 8. Let $L : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be the linear transformation defined by

$$L(\mathbf{x}) = \begin{pmatrix} x_1 \\ x_1 + x_2 \\ -x_1 + x_2 \end{pmatrix}$$

and let $E = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $F = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$, where

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \mathbf{b}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{b}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

The matrix representing L with respect to the ordered bases E and F is given by

- (a) $\begin{pmatrix} 1 & 3 \\ 3 & 4 \\ 1 & -2 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} -2 & 2 & 1 \\ -1 & 6 & -2 \end{pmatrix}$
(d) $\begin{pmatrix} -2 & -1 \\ 2 & 6 \\ 1 & 2 \end{pmatrix}$ (e) $\begin{pmatrix} -2 & -1 \\ 2 & 6 \\ 1 & -2 \end{pmatrix}$

Exercise 9. Let L be the linear operator mapping \mathbb{R}^3 into \mathbb{R}^3 defined by $L(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}$$

The matrix representing L with respect to the ordered basis $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$, where

$$\mathbf{y}_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{y}_2 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \quad \mathbf{y}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

is given by

(a) $\begin{pmatrix} -3 & 6 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & -2 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & -1 & 0 \\ 2/3 & -1/3 & -1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$

(e) $\begin{pmatrix} -3 & 6 & 0 \\ -2 & 4 & 0 \\ 1 & 0 & 4 \end{pmatrix}$

Exercise 10. Let $\mathbf{x} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \end{pmatrix}$.

Let d be the distance and θ the angle between \mathbf{x} and \mathbf{y} . Then,

$$d^2 + \tan \theta =$$

(a) 3

(b) 4

(c) 5

(d) 6

(e) 7

Exercise 11. Recall that the trace of a square matrix is the sum of its diagonal entries. Let A be an $n \times n$ matrix and let $\|A\|_F$ denote its Frobenius norm. Then,

$$\|A\|_F^2 =$$

- (a) $\text{trace}(A^2)$
- (b) $\text{trace}(A^T)$
- (c) $\text{trace}(A)\text{trace}(A^T)$
- (d) $\text{trace}(A^T A)$
- (e) $(\text{trace}(A) + \text{trace}(A^T))^2$

Exercise 12. Let S be the set of all vectors in \mathbb{R}^3 of the form $\begin{pmatrix} x \\ y \\ x \end{pmatrix}$ and consider the

vector $\mathbf{w} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$ in \mathbb{R}^3 . Then, the vector $\mathbf{p} \in S$ that is closest to \mathbf{w} is given by

- (a) $\mathbf{p} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ (b) $\mathbf{p} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$ (c) $\mathbf{p} = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$ (d) $\mathbf{p} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ (e) $\mathbf{p} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

Exercise 13. Consider the vector space $C[-1,1]$ with the inner product defined by

$$\langle f, g \rangle = \int_{-1}^1 f(x)g(x) dx.$$

An *orthonormal* basis for the subspace spanned by $1, x, x^2$ is given by

- (a) $1, x, x^2 - \frac{1}{3}$
- (b) $\frac{1}{\sqrt{2}}, x\sqrt{\frac{3}{2}}, (x^2 - \frac{1}{3})$
- (c) $\frac{1}{\sqrt{2}}, x\sqrt{\frac{3}{2}}, (3x^2 - 1)\sqrt{\frac{5}{8}}$
- (d) $\frac{1}{\sqrt{2}}, x, (3x^2 - 1)\sqrt{\frac{5}{8}}$
- (e) $1, x\sqrt{\frac{3}{2}}, (3x^2 - 1)\sqrt{\frac{5}{8}}$

Exercise 14. Let $A = \begin{pmatrix} 3 & 1 & 2 \\ 0 & 1 & -2 \\ 0 & 1 & 4 \end{pmatrix}$. Then:

- (a) A has no real eigenvalues
- (b) A has two distinct eigenvalues λ_1 and λ_2 with $\dim(\text{Eigenspace}(\lambda_1)) = 1$ and $\dim(\text{Eigenspace}(\lambda_2)) = 1$
- (c) A has two distinct eigenvalues λ_1 and λ_2 with $\dim(\text{Eigenspace}(\lambda_1)) = 1$ and $\dim(\text{Eigenspace}(\lambda_2)) = 2$
- (d) A has one eigenvalue λ with $\dim(\text{Eigenspace}(\lambda)) = 3$
- (e) A has three distinct eigenvalues and each has an eigenspace of dimension 1

Exercise 15. Let A be an $n \times n$ nonsingular matrix and let λ be an eigenvalue of A . Then:

- (a) λ is an eigenvalue of A^{-1}
- (b) $\frac{1}{\lambda}$ is an eigenvalue of A^T
- (c) $\lambda - 1$ is an eigenvalue of $A - I$
- (d) $\lambda(\lambda - 1) = 0$
- (e) λ is an eigenvalue of A^2

Exercise 16. Let $A = \begin{pmatrix} 3 & -1 & 2 \\ 2 & 0 & 2 \\ 2 & -1 & 3 \end{pmatrix}$. If we factor A into a product DXD^{-1} , where D is diagonal, then $X =$

- (a) $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$
- (b) $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}$
- (c) $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}$
- (d) $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$
- (e) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & -1 & 1 \end{pmatrix}$

Exercise 17. Let A be a *diagonalizable* 7×7 matrix. If A has only two distinct eigenvalues λ_1 and λ_2 such that $\lambda_1 I - A$ has rank 3, then the multiplicity of λ_2 is equal to

- (a) 1 (b) 2 (c) 3 (d) 4 (e) 5

Exercise 18. Consider the conic section $2x^2 + 2xy + 2y^2 - 9 = 0$. If a standard form for this quadratic equation is given by

$$\frac{x'^2}{a} + \frac{y'^2}{b} = 1$$

then $a^2 + b^2 =$

- (a) 10 (b) 18 (c) 82 (d) 90 (e) 162

Exercise 19. Consider the conic section $3x^2 + 2xy + 3y^2 + 8\sqrt{2}x - 8 = 0$. If a standard form for this quadratic equation is given by

$$\frac{x'^2}{a} + \frac{y'^2}{b} = 1$$

then $a + b =$

(a) 14

(b) 15

(c) 16

(d) 17

(e) 18

Exercise 20. Find the symmetric matrix associated with the quadratic form

$$x^2 + 9y^2 - 7z^2 + xy - 2xz - yz$$

(a) $\begin{pmatrix} 1 & 1 & -2 \\ 1 & 9 & -1 \\ -2 & -1 & -7 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 1 & -1 \\ 1 & 9 & -1 \\ -1 & -1 & -7 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & \frac{1}{2} & -1 \\ \frac{1}{2} & 9 & -\frac{1}{2} \\ -1 & -\frac{1}{2} & -7 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & \frac{1}{2} & -2 \\ \frac{1}{2} & 9 & -1 \\ -2 & -1 & -7 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 9 & 0 \\ -1 & 0 & -7 \end{pmatrix}$