01

King Fahd University of Petroleum & Minerals
Department of Mathematics
Math 302 Major Exam I
The First Semester of 2021-2022 (211)
Time Allowed: 120 Minutes

Name: Solution ID#:

Section/Instructor: Key- Serial #:

- Mobiles and calculators are not allowed in this exam.
- Write neatly and eligibly. You may lose points for messy work.
- Show all your work. No points for answers without justification.

Question #	Marks	Maximum Marks
1		12
2		08
3		12
4		14
5		17
6		12
Total		75

Q:1 (8 + 4 = 12 points) (a) Prove or disprove
(i) The set of real numbers with addition defined as

$$x + y = x - y \quad for\ all\ x, y \in \mathbb{R}$$

is a vector space.
(ii) The set $S = \{(a, b) | b = 2a\}$ is a subspace of $\mathbb{R}^2$ (under the usual addition and scalar multiplication).
(b) Is the vector $< 1, 1, 1, 1 >$ a linear combination of vectors $< 1, 0, 1, 1 >$ and $< 1, 1, 0, 1 >$? **(Justify your answer)**.

a (i) $\mathbb{R}$ is not a vector space with this operation, since it is not associative. For escample

$$2-(3-5) = 2-(-2) = 4$$
$$(2-3)-5 = -1-5 = -6$$

(Not equal)

(ii) S is a subspace of $\mathbb{R}^2$, since it is closed and scalar multiplication

- For any $(a,b), (c,d) \in S$, we have $(a,b) + (c,d) = (a+c, b+d) \in S$
  since $b+d = 2a + 2c = 2(a+c)$
- For any $k \in \mathbb{R}$ and $(a,b) \in S$, we have
  $k(a,b) = (ka, kb) \in S$
  since $kb = k(2a) = 2(ka)$.

(b) $<1,1,1,1> = \alpha <1,0,1,1> + \beta <1,1,0,1>$, $\alpha, \beta \in R$

$$\Rightarrow \quad \alpha + \beta = 1$$
$$\alpha = 1$$
$$\beta = 1$$

$\Rightarrow$ System is inconsistent
$\Rightarrow$ Vector $<1,1,1,1>$ can not be written as a linear combination of $<1,0,1,1>$ and $<1,1,0,1>$.

Q:2 (08 points) Solve the following nonhomogeneous system using **Gaussian Elimination method:**

$$\begin{aligned} x_1 - x_2 - 2x_3 &= -1 \\ -3x_1 - 2x_2 + x_3 &= -7 \\ 2x_1 + 3x_2 + x_3 &= 8 \end{aligned}$$

Sol:

Augmented matrix $= \left(\begin{array}{ccc|c} 1 & -1 & -2 & -1 \\ -3 & -2 & 1 & -7 \\ 2 & 3 & 1 & 8 \end{array}\right)$

$\underset{R_3-2R_1}{\overset{R_2+3R_1}{\sim}} \left(\begin{array}{ccc|c} 1 & -1 & -2 & -1 \\ 0 & -5 & -5 & -10 \\ 0 & 5 & 5 & 10 \end{array}\right)$

$\overset{\frac{R_2}{-5}}{\sim} \left(\begin{array}{ccc|c} 1 & -1 & -2 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 5 & 5 & 10 \end{array}\right)$

$\overset{R_3-5R_2}{\sim} \left(\begin{array}{ccc|c} 1 & -1 & -2 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\Rightarrow \; x_1 - x_2 - 2x_3 = -1 \;,\; x_2 + x_3 = 2.$

Let $x_3 = t \in \mathbb{R}$ Then $x_2 = 2 - t$ and $x_1 = 1 + t$.

System has infinitely many solutions.

Q3 (12 points) Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 1 & -2 & 4 \\ 1 & 2 & -3 \end{pmatrix},$$

(i) Find the rank of the matrix A.
(ii) Find a basis for the row space of A.

(i) $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 1 & -2 & 4 \\ 1 & 2 & -3 \end{pmatrix}$

$\underset{\substack{R_3 - R_1 \\ R_4 - R_1}}{\overset{-R_2}{\sim}} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & -4 & 1 \\ 0 & 0 & -6 \end{pmatrix}$

$\overset{R_3 + 4R_2}{\sim} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & -7 \\ 0 & 0 & -6 \end{pmatrix}$. $\underset{-R_4/6}{\overset{-R_3/7}{\sim}} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

$\overset{R_4 - R_3}{\sim} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$\overset{R_1 - 2R_2}{\sim} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\overset{R_1 - R_3}{\sim} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$\overset{R_2 + 2R_3}{\sim} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Rank of A is 3

(ii) A possible basis for the row space of A is
$B = \{(1,0,0), (0,1,0), (0,1,1)\}$.
Any basis of $\mathbb{R}^3$ works (for example the first three rows of A)

Q4: (14 points) Consider the system of linear equations $AX = B$, where

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 0 & -1 & 2 \\ 1 & 4 & -3 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ and } \quad B = \begin{pmatrix} 6 \\ 3 \\ -6 \end{pmatrix}.$$

(a) Show that the matrix A is invertible and find A^{-1}.

(b) Solve the system of linear equations $AX = B$.

$|A| \neq 0 \Rightarrow A$ is invertible.

Augmented matrix $(A \vdots I) = \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 1 & 4 & -3 & 0 & 0 & 1 \end{array}\right)$

$\overset{R_3-R_1}{\sim} \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 0 & 6 & -6 & -1 & 0 & 1 \end{array}\right)$

$\overset{R_3/6}{\sim} \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & -1 & 0 \\ 0 & 1 & -1 & -\frac{1}{6} & 0 & \frac{1}{6} \end{array}\right) \overset{R_3-R_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{6} & 1 & \frac{1}{6} \end{array}\right)$

$\overset{R_2+2R_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{3} & 1 & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{6} & 1 & \frac{1}{6} \end{array}\right)$

$\overset{R_1+2R_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & \frac{1}{3} & 2 & \frac{2}{3} \\ 0 & 1 & 0 & -\frac{1}{3} & 1 & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{6} & 1 & \frac{1}{6} \end{array}\right)$

$\overset{R_1-3R_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{6} & -1 & \frac{1}{6} \\ 0 & 1 & 0 & -\frac{1}{3} & 1 & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{6} & 1 & \frac{1}{6} \end{array}\right)$

$A^{-1} = \begin{pmatrix} \frac{5}{6} & -1 & \frac{1}{6} \\ -\frac{1}{3} & 1 & \frac{1}{3} \\ -\frac{1}{6} & 1 & \frac{1}{6} \end{pmatrix}$

(b) The solution of the system $AX = B$ is $X = A^{-1}B$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{5}{6} & -1 & \frac{1}{6} \\ -\frac{1}{3} & 1 & \frac{1}{3} \\ -\frac{1}{6} & 1 & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 6 \\ 3 \\ -6 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Q:5 (3 + 10 + 4 = 17 points) Let A = $\begin{pmatrix} 1 & 0 & \sqrt{2} \\ 0 & 2 & 0 \\ \sqrt{2} & 0 & 0 \end{pmatrix}$.

(i) Is matrix A orthogonally diagonalizable? **(Justify your answer)**
(ii) Find an orthogonal matrix P that diagonalizes A and the digonal matrix $D = P^T AP$.
(iii) Compute A^8.

(i) Yes, any ~~real~~ symmetric matrix is orthogonally diagonalizable.

(ii) $|A-\lambda I| = \begin{vmatrix} 1-\lambda & 0 & \sqrt{2} \\ 0 & 2-\lambda & 0 \\ \sqrt{2} & 0 & -\lambda \end{vmatrix} = 0 \Rightarrow (\lambda+1)(\lambda-2)^2 = 0$

$\Rightarrow \lambda = -1, \lambda = \sqrt{2}, \lambda = -\sqrt{2}$

or $\lambda = -1, 2, 2$

A possible set of eigenvectors of A is $S = \{u, v, w\}$,

where $u = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ 1 \end{pmatrix}$ for $\lambda = -1$

$v = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \sqrt{2} \\ 0 \\ 1 \end{pmatrix}$ for $\lambda = 2$

Notice that S is an orthogonal set.

We have $\|u\| = \sqrt{(-\frac{1}{\sqrt{2}})^2 + 0 + 1} = \frac{\sqrt{3}}{\sqrt{2}}$

$\|v\| = 1$

$\|w\| = \sqrt{3}$

Divide each vector by its norm, we obtain the orthogonal set

$\left\{\left(-\frac{1}{\sqrt{3}}, 0, \frac{\sqrt{2}}{\sqrt{3}}\right), (0,1,0), \left(\frac{\sqrt{2}}{\sqrt{3}}, 0\ \frac{1}{\sqrt{3}}\right)\right\}$

Orthogonal matrix $P = \begin{pmatrix} -\frac{1}{\sqrt{3}} & 0 & \frac{\sqrt{2}}{\sqrt{3}} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}$

$D = P^{-1}AP$; $P^T AP = \begin{pmatrix} -\frac{1}{\sqrt{3}} & 0 & \frac{\sqrt{2}}{\sqrt{3}} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 1 & 0 & \sqrt{2} \\ 0 & 2 & 0 \\ \sqrt{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{3}} & 0 & \frac{\sqrt{2}}{\sqrt{3}} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}$

$(P^{-1} = P)$

$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

(iii) $A^8 = P\, D^{10}\, P^T$

$$= \begin{pmatrix} -\frac{1}{\sqrt{3}} & 0 & \frac{\sqrt{2}}{\sqrt{3}} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 256 & 0 \\ 0 & 0 & 256 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{3}} & 0 & \frac{\sqrt{2}}{\sqrt{3}} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{\sqrt{3}} & 0 & \frac{\sqrt{2}}{\sqrt{3}} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{3}} & 0 & \frac{\sqrt{2}}{\sqrt{3}} \\ 0 & 256 & 0 \\ \frac{256\sqrt{2}}{\sqrt{3}} & 0 & \frac{256}{\sqrt{3}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3} + \frac{256\times 2}{3} & 0 & -\frac{\sqrt{2}}{3} + \frac{256\sqrt{2}}{3} \\ 0 & 256 & 0 \\ -\frac{\sqrt{2}}{3} + \frac{256\times 2}{3} & 0 & \frac{2}{3} + \frac{256}{3} \end{pmatrix}$$

Q:6 (19 points) Let $A = \begin{pmatrix} \sqrt{2}\cos x & i\sin x & 0 \\ i\sin x & 0 & -i\sin x \\ 0 & -i\sin x & -\sqrt{2}\cos x \end{pmatrix}$.

(i) Find eigenvalues of the matrix A.
(ii) For what values of x $(0 \le x \le \frac{\pi}{2})$ is the matrix A diagonalizable $(i^2 = -1)$.

(i) $$|A - \lambda I| = \begin{vmatrix} \sqrt{2}\cos x - \lambda & i\sin x & 0 \\ i\sin x & -\lambda & -i\sin x \\ 0 & -i\sin x & -\sqrt{2}\cos x - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (\sqrt{2}\cos x - \lambda)\left[\lambda^2 + \sqrt{2}\lambda\cos x + \sin^2 x\right] - i\sin x\left[-i\sin x\sqrt{2}\cos x - i\lambda\sin x\right] = 0$$

$$-\lambda^3 - \sqrt{2}\lambda^2\cos x - \lambda\sin^2 x + \sqrt{2}\lambda^2\cos x + 2\lambda\cos^2 x + \sqrt{2}\cos x\sin^2 x$$
$$-\sqrt{2}\sin^2 x\cos x - \lambda\sin^2 x = 0$$

$$\Rightarrow \quad -\lambda^3 + 2\lambda(\cos^2 x - \sin^2 x) = 0$$
$$-\lambda(\lambda^2 - 2\cos 2x) = 0 \quad \Rightarrow \lambda = 0, \pm\sqrt{2\cos 2x}$$

If $\sqrt{2\cos 2x} = 0$, then we have $\cos 2x = 0$
$$\Rightarrow x = \frac{\pi}{4}, \cancel{\frac{3\pi}{4}}, \quad (0 \le x \le \pi/2)$$

If $x \ne \frac{\pi}{4}$, then matrix has three ~~dis~~ distinct eigenvalues

$\Rightarrow$ A is diagonalizable.

If $x = \pi/4$, then $\lambda = 0, 0, 0$,

For $\lambda = 0$, $x = \pi/4$: $(A - \lambda I \,\vdots\, 0) = \left(\begin{array}{ccc|c} 1 & \frac{i}{\sqrt{2}} & 0 & 0 \\ \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} & 0 \\ 0 & -\frac{i}{\sqrt{2}} & -1 & 0 \end{array}\right)$

$$\Rightarrow k_1 = -\frac{i}{\sqrt{2}}k_2 \;;\; k_1 = k_3,\; k_3 = -\frac{i}{\sqrt{2}}k_2$$

Let $k_1 = k_2 = 1$. Then $K = \begin{pmatrix} 1 \\ i\sqrt{2} \\ 1 \end{pmatrix}$

or $K = \begin{pmatrix} i \\ -\sqrt{2} \\ i \end{pmatrix}$ or $\begin{pmatrix} -i \\ \sqrt{2} \\ -i \end{pmatrix}$

It is not possible to find three L.I. eigenvectors.
Hence matrix A is not diagonalizable.