001

King Fahd University of Petroleum & Minerals
Department of Mathematics
Math 302 Major Exam II
The First Semester of 2021-2022 (211)

Time Allowed: 120 Minutes

Name: ______________________ ID#: ______________

Section/Instructor: Key Serial #: ______________

- Mobiles, calculators and smart devices are not allowed in this exam.
- Write neatly and eligibly. You may lose points for messy work.
- Show all your work. **No points for answers without justification.**

Question #	Marks	Maximum Marks
1		10
2		12
3		10
4		10
5		13
6		10
7		10
Total		75

Q:1 (10 points) Let $\widehat{A} = \widehat{a_x} - \widehat{a_y} + 3\ \widehat{a_z}$ and $\widehat{B} = -2\widehat{a_\rho} - \widehat{a_\phi} + 3\ \widehat{a_z}$.
(i) Transform $\widehat{B}$ into Cartesian coordinates.
(ii) Find $\widehat{A} - \widehat{B}$ at $P(x, y, z) = P(0, 3, -1)$
(iii) Compute the angle between $\widehat{A}$ and $\widehat{B}$ at P.
(iv) Find the scalar component of $\widehat{A}$ along $\widehat{B}$ at P.

(i) $\hat{B} = (-2\cos\phi + \sin\phi)\hat{a_x} + (-2\sin\phi + \cos\phi)\hat{a_y} + 3\hat{a_z}$

$= \left(-\frac{2x}{\sqrt{x^2+y^2}} + \frac{y}{\sqrt{x^2+y^2}}\right)\hat{a_x} + \left(-\frac{2y}{\sqrt{x^2+y^2}} + \frac{x}{\sqrt{x^2+y^2}}\right)\hat{a_y} + 3\hat{a_z}$

(ii) $\hat{B}$ at $P(0,3,-1)$ is

$\hat{B} = \hat{a_x} - 2\hat{a_y} + 3\hat{a_z}$

$\hat{A} - \hat{B} = \hat{a_y}$

OR

P is $(3, \frac{\pi}{2}, -1)$ in cylindrical coordinates

$\hat{B} = \left(-2\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right)\hat{a_x} + \left(-2\sin\frac{\pi}{2} + \cos\frac{\pi}{2}\right)\hat{a_y} + 3\hat{a_z}$

$= \hat{a_x} - 2\hat{a_y} + 3\hat{a_z}$

$\hat{A} - \hat{B} = \hat{a_y}$

(iii) $\alpha = \cos^{-1}\left(\frac{\hat{A}\cdot\hat{B}}{|\hat{A}|\,|\hat{B}|}\right) = \cos^{-1}\left(\frac{12}{\sqrt{11}\ \sqrt{14}}\right)$

(iv) The scalar component of $\hat{A}$ along $\hat{B}$ at P is

$|\hat{A}|\cos\alpha$

$= \frac{\hat{A}\cdot\hat{B}}{|\hat{B}|} = \frac{12}{\sqrt{14}}$

Q:2 (8 + 4 = 12 points) (a) Let $V(x, y, z) = x^3y^2\sin(3z)$.
(i) Find ∇V at $P(x, y, z) = P(1, -1, 0)$.
(ii) Calculate the directional derivative of V at P in the direction of $Q(x, y, z) = Q(2, 0, \pi)$.

(i) $\nabla V = 3x^2y^2\sin(3z)\,\hat{a}_x + 2x^3y\sin(3z)\,\hat{a}_y + 3x^3y^2\cos(3z)\,\hat{a}_z$

$\nabla V(1,-1,0) = 3\,\hat{a}_z$

(ii) $\hat{PQ} = \langle 2, 0, \pi\rangle - \langle 1, -1, 0\rangle$

$= \langle 1, 1, \pi\rangle.$

$D_{\hat{PQ}}V = \hat{\nabla V_P} \cdot \dfrac{\hat{PQ}}{|\hat{PQ}|}$

$= \dfrac{3\pi}{\sqrt{2+\pi^2}}$

(b) Let $\widehat{A} = x^2y\,\widehat{a_x} - y\,\widehat{a_y}$. Find $\int_L \widehat{A}\bullet\widehat{dl}$, where L is the line joining $(1, 1)$ and $(2, 0)$.

Equation of line joining $(1,1)$ and $(2,0)$ is

$y = 2 - x$

$dy = -dx$

$\int_L \hat{A}\cdot\hat{dl} = \int_L (x^2y\,dx - y\,dy)$

$= \int_{x=1}^{2} \left[x^2(2-x)\,dx - (2-x)(-dx)\right]$

$= \int_1^2 (-x^3 + 2x^2 - x + 2)\,dx$

$= -\dfrac{x^4}{4} + \dfrac{2x^3}{3} - \dfrac{x^2}{2} + 2x \Big|_1^2$

$= \dfrac{17}{12}$

Q:3 (10 points) Use **Stokes's Theorem** to evaluate $\oint_C (z^2\, e^{x^2})\, dx + (xy^2)\, dy + (\tan^{-1} y)\, dz$, where C is the circle defined by $x^2 + y^2 = 9$ (orientation of C is counter-clockwise as viewed from above).

Sol: $\hat{E} = z^2 e^{x^2}\, \hat{a}_x + xy^2\, \hat{a}_y + \tan^{-1} y\, \hat{a}_z, \; d\hat{s} = ds\, \hat{a}_z$

$$\nabla \times \hat{E} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 e^{x^2} & xy^2 & \tan^{-1} y \end{vmatrix}$$

$$= \left(\frac{1}{1+y^2} - 0\right)\hat{a}_x - \left(0 - 2z e^{x^2}\right)\hat{a}_y + (y^2 - 0)\hat{a}_z$$

Stokes' theorem $\oint_C \hat{E} \cdot d\hat{l} = \iint_S (\nabla \times \hat{E}) \cdot d\hat{s}$

$$= \iint_R y^2\, ds$$

$$= \int_{\phi=0}^{2\pi} \int_{\rho=0}^{3} \rho^2 \sin^2\phi \cdot \rho\, d\rho\, d\phi$$

$$= \frac{\rho^4}{4}\Big|_0^3 \int_{\phi=0}^{2\pi} \left(\frac{1-\cos 2\phi}{2}\right) d\phi$$

$$= \frac{81}{4}\left[\frac{\phi}{2} - \frac{\sin 2\phi}{4}\right]_0^{2\pi}$$

$$= \frac{81}{4}\left[\pi - 0 - 0 + 0\right]$$

$$= \frac{81\pi}{4}$$

Q:4 (10 points) Let D be the region described by the spherical shell $1 < r < 2$, $\frac{\pi}{3} < \theta < \frac{\pi}{2}$, $\frac{\pi}{4} < \phi < \frac{\pi}{3}$. Use the **Divergence theorem** to find the **outward flux** of the vector field $\widehat{A} = r\cos\phi\ \widehat{a_r} + r\sin\theta\ \widehat{a_\theta} + r^3\ \widehat{a_\phi}$.

Divergence theorem $\quad \int_S \hat{E}\cdot d\hat{S} = \int_D \nabla\cdot\hat{E}\, dV$

$$\nabla\cdot\hat{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot r\cos\phi\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \cdot r\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(r^3\right)$$

$$= 3\cos\phi + \frac{1}{r\sin\theta}\cdot r\cdot 2\sin\theta\cos\theta + 0$$

$$= 3\cos\phi + 2\cos\theta$$

$dV = r^2\sin\theta\, dr\, d\theta\, d\phi$

$$\int_S \hat{E}\cdot d\hat{S} = \int_{\phi=\pi/4}^{\pi/3}\int_{\theta=\pi/3}^{\pi/2}\int_{r=1}^{2}\left(3\cos\phi + 2\cos\theta\right) r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \frac{7}{3}\int_{\pi/4}^{\pi/3}\int_{\pi/3}^{\pi/2}\left(3\cos\phi\sin\theta + \sin 2\theta\right) d\theta\, d\phi$$

$$= \frac{7}{3}\left[3\left(\sin\phi\right)\Big|_{\pi/4}^{\pi/3}\left(-\cos\theta\right)\Big|_{\pi/3}^{\pi/2} + \left(-\frac{\cos 2\theta}{2}\right)\Big|_{\pi/3}^{\pi/2}\left(\phi\right)\Big|_{\pi/4}^{\pi/3}\right]$$

$$= \frac{7}{3}\left[3\left(\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\right)\left(\frac{1}{2} - 0\right) + \frac{\pi}{24}\left(+1 + \frac{1}{2}\right)\right]$$

$$= \frac{7}{3}\left[\frac{3}{4}\left(\sqrt{3} - \sqrt{2}\right) + \frac{\pi}{48}\right]$$

Q:5 (13 points) Let $\widehat{E} = 4\,\rho\,\sin\phi\,\widehat{a}_\rho + 2\,\rho\,\cos\phi\,\widehat{a}_\phi + 2\,z^2\,\widehat{a}_z$ be the electric field on a certain region of space.

(a) Verify that $\widehat{E}$ is a conservative field.

(b) Find the electric potential V.

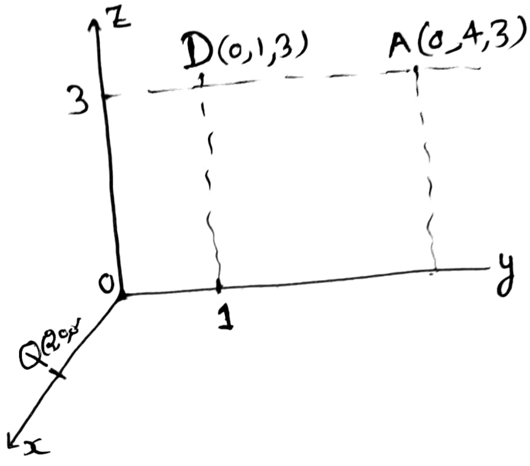
(c) Evaluate $\int_{(1,0,-2)}^{(4,\frac{\pi}{2},1)} \widehat{E}.\widehat{dl}$. (indicates all points are given in cylindrical coordinates)

(a)
$$\nabla \times \hat{E} = \frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\,\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 4\rho\sin\phi & 2\rho^2\cos\phi & 2z^2 \end{vmatrix}$$
$$= \frac{1}{\rho}\left[(0-0)\hat{a}_\rho - \rho(0-0)\hat{a}_\phi + (4\rho\cos\phi - 4\rho\cos\phi)\hat{a}_z\right]$$
$$= \hat{0} \quad \Rightarrow \quad \hat{E} \text{ is a conservative field.}$$

(b)
$$\hat{E} = -\nabla V \Leftrightarrow \hat{E} = -\left\langle \frac{\partial V}{\partial \rho}, \frac{1}{\rho}\frac{\partial V}{\partial \phi}, \frac{\partial V}{\partial z}\right\rangle$$
$$\frac{\partial V}{\partial \rho} = -4\rho\sin\phi, \quad \frac{1}{\rho}\frac{\partial V}{\partial \phi} = -2\rho\cos\phi, \quad \frac{\partial V}{\partial z} = -2z^2$$
$$V = -2\rho^2\sin\phi + G(\phi, z)$$
$$\frac{\partial V}{\partial \phi} = -2\rho^2\cos\phi + G_\phi(\phi, z) = -2\rho^2\cos\phi$$
$$\Rightarrow G_\phi(\phi,z) = 0 \Rightarrow G(\phi,z) = H(z)$$
$$\therefore V = -2\rho^2\sin\phi + H(z)$$
$$\frac{\partial V}{\partial z} = 0 + H'(z) = -2z^2$$
$$\Rightarrow H(z) = -\frac{2}{3}z^3 + C$$
$$\boxed{V = -2\rho^2\sin\phi - \frac{2}{3}z^3 + C}$$

(c)
$$V(1,0,-2) = \frac{16}{3} + C$$
$$V(4,\tfrac{\pi}{2},1) = (-2)(16)(1) - \frac{2}{3} + C = -\frac{98}{3} + C$$
$$\int_{(1,0,-2)}^{(4,\pi/2,1)} \hat{E}\cdot d\hat{l} = -\frac{98}{3} - \frac{16}{3} + C - C$$
$$= -\frac{114}{3}$$
$$= -38$$

Q:6 (10 points) A point charge of 4 μC is located at $(2,0,0)$, while the line $y=1$, $z=3$ carries an uniform charge of 7 μC. If the potential at $A(0,4,3)$ is V = 17 , find the potential at $(0,0,0)$. (all points are in cartesian coordinates)



$$V = V_Q + V_L$$

$$= \frac{Q}{4\pi\epsilon_0 r} - \frac{\rho_L}{2\pi\epsilon_0}\ln\rho$$

$$r_0 = \left|(0,0,0)-(2,0,0)\right| = 2$$

$$\rho_0 = \left|(0,0,0)-(0,1,3)\right| = \sqrt{10}$$

$$r_A = \left|(0,4,3)-(2,0,0)\right| = \sqrt{29}$$

$$\rho_A = \left|(0,4,3)-(0,1,3)\right| = 3$$

Hence $V_0 - V_A = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r_0} - \frac{1}{r_A}\right] - \frac{\rho_L}{2\pi\epsilon_0}\ln\left(\frac{\rho_0}{\rho_A}\right)$

$$V_0 = 17 + \frac{4}{4\pi\epsilon_0}\left[\frac{1}{2} - \frac{1}{\sqrt{29}}\right] - \frac{7}{2\pi\epsilon_0}\ln\left(\frac{\sqrt{10}}{3}\right)$$

Q:7 (4 + 6 =10 points) (a) Find all solutions of the equation $z^4 + 1 = 0$.

$$z = (-1)^{1/4} \quad ; \quad r = 1,\ \theta = \pi,\ n = 4$$

$$w_k = \cos\left(\frac{\pi + 2k\pi}{4}\right) + i \sin\left(\frac{\pi + 2k\pi}{4}\right),\ k = 0, 1, 2, 3$$

$$w_0 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$$

$$w_1 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$$

$$w_2 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i$$

$$w_3 = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i$$

(b) Sketch the complex region that describes the intersection between $|z - 3 - 4i| < 5$ and $0 \le Im(z) \le 4$.

$|z-3-4i| < 5$ is an open disk with center $(3,4)$ and radius 5

$0 \le Im(z) \le 4$ is the region bounded by $y = 0$ and $y = 4$