FE

King Fahd University of Petroleum & Minerals
Department of Mathematics
Math 302 Final Exam
The First Semester of 2021-2022 (211)
Time Allowed: 150 Minutes

---

Name: ______________________ ID#: ______________

Section/Instructor: ______________________ Serial #: ______________

---

- Smart devices and calculators are not allowed in this exam.
- Write neatly and eligibly. You may lose points for messy work.
- **Show all your work. No points for answers without justification.**

---

| Question # | Marks | Maximum Marks |
|---|---|---|
| 1 | | 10 |
| 2 | | 10 |
| 3 | | 10 |
| 4 | | 10 |
| 5 | | 10 |
| 6 | | 10 |
| 7 | | 15 |
| 8 | | 15 |
| 9 | | 15 |
| Total | | 105 |

**Q:1** (10 points) Find a matrix **P** that diagonalizes

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

Sol:

$$|A - \lambda I| = 0 = (\lambda+1)(\lambda-1)(\lambda-3)$$

$$\Rightarrow \lambda = -1, 1, 3$$

The eigenvectors of A are

$$\left\{\begin{matrix} -1 \\ 0 \\ 1 \end{matrix}\right\} \leftrightarrow -1, \quad \left\{\begin{matrix} 0 \\ 1 \\ 0 \end{matrix}\right\} \leftrightarrow 1, \quad \left\{\begin{matrix} 1 \\ 0 \\ 1 \end{matrix}\right\} \leftrightarrow 3$$

We can choose

$$P = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

**Q:2** (10 points) Let $\widehat{F} = \frac{1}{3}z^3\,\widehat{k}$. Use the divergence theorem to evaluate $\oint_S \widehat{F}.\widehat{dS}$, where $S$ is the unit sphere.
(Hint: $\oint_S \widehat{F}.\widehat{dS} = \int_V \nabla.\widehat{F}\ dV$, $x = r\ \sin\theta\cos\phi$, $y = r\ \sin\theta\sin\phi$, $z = r\ \cos\theta$)

Solution: Using the divergence theorem on the region R given by the unit ball, we have

$$\iint_S \hat{F}\cdot d\hat{A} = \iiint_R (\hat{\nabla}\cdot\hat{F})\,dV$$

$$= \iiint_R z^2\,dV$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{1} (r^2\cos^2\theta)\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \frac{4}{15}\pi.$$

**Q:3** (2 + 6 + 2 points) Consider $u(x, y) = e^x \sin(y) + 2y$.
(a) Verify that $u(x, y)$ is a harmonic function.
(b) Find a harmonic conjugate $v(x, y)$ of $u(x, y)$.
(c) Use $u(x, y)$ and $v(x, y)$ to construct an entire function $f(z)$.

Sol: (a) We have

$$u_x = e^x \sin y, \quad u_{xx} = e^x \sin y, \quad u_y = e^x \cos y + 2, \quad u_{yy} = -e^x \sin y$$

$$u_{xx} + u_{yy} = e^x \sin y - e^x \sin y = 0$$

(b) Let $v(x,y)$ be a harmonic conjugate of $u(x,y)$.
Since $u$ and $v$ satisfy the Cauchy-Riemann equations, we have

$$v_y = u_x = e^x \sin y$$

$$\Rightarrow v(x,y) = -e^x \cos y + g(x)$$

Since $u$ and $v$ satisfy the Cauchy-Riemann equations, we have

$$u_y = -v_x$$

$$e^x \cos y + 2 = e^x \cos y - g'(x)$$

$$\Rightarrow g'(x) = -2$$

$$g(x) = -2x + C$$

$$\therefore v(x,y) = -e^x \cos y - 2x + C$$

is a harmonic conjugate of $u(x,y)$, where $C$ is any real number.

(C) The following function

$$f(z) = u(x,y) + i\, v(x,y)$$

$$= e^x \sin y + 2y - i\left(e^x \cos y + 2x - C\right)$$

is an entire function, where $C$ is any real number.

**Q:4** (10 points) If $W = \coth z = \dfrac{\cosh z}{\sinh z}$, show that $z = \dfrac{1}{2}\ln(\dfrac{W+1}{W-1})$, $W \neq \pm 1$

$$\coth z = \frac{(e^{z} + e^{-z})/2}{(e^{z} - e^{-z})/2} = W$$

$$W(e^{z} - e^{-z}) = e^{z} + e^{-z}$$

$$\Rightarrow W(1 - e^{-2z}) = 1 + e^{-2z}$$

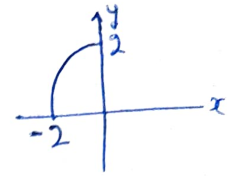
$$(W-1) = (1+W)e^{-2z}$$

$$e^{2z} = \frac{W+1}{W-1}$$

$$2z = \ln\left(\frac{W+1}{W-1}\right), \quad W \neq \pm 1$$

$$z = \frac{1}{2}\ln\left(\frac{W+1}{W-1}\right), \quad W \neq \pm 1$$

**Q:5** (10 points) Evaluate $\oint_C Re(z-1)\, dz$, where $C$ is the circular arc in the second quadrant along $|z| = 2$ from $z = 2i$ to $z = -2$.




Sol:

$$\int_C Re(Z-1)\, dZ.$$

$$|Z| = 2 \Leftrightarrow Z = 2e^{it}$$

$$= \int_{\frac{\pi}{2}}^{\pi} (2\cos t - 1) \cdot 2i\, e^{it}\, dt$$

$$= 2i \int_{\frac{\pi}{2}}^{\pi} \left(e^{it} + e^{-it} - 1\right) e^{it}\, dt$$

$$= 2i \int_{\pi/2}^{\pi} \left(e^{2it} - e^{it} + 1\right) dt$$

$$= e^{2it} - 2e^{it} + 2it \Big|_{\pi/2}^{\pi}$$

$$= (1 + 2 + 2i\pi) - (-1 - 2i + i\pi)$$

$$= 4 + i\pi + 2i = 4 + i(\pi + 2)$$

**Q:6** (10 points) Let $C$ be the path consisting of the sides of the triangle in the $xy-$ plane with vertices $(0,0)$, $(2,0)$ and $(1,2)$ counter-clockwise
Find

(a) $\oint_C \frac{dz}{z-(1+i)}$

(b) $\oint_C \frac{dz}{(z-(1+i))^{10}}$

Sol. Applying Deformation of contours, we can replace C with any circle centered at $(1,1)$ with radius small enough that it lies inside the triangle directed counterclockwise

Applying the Cauchy-Goursat Theorem, we have

$$\oint \frac{dz}{z-(1+i)} = 2\pi i$$

$$\oint \frac{dz}{(z-(1+i))^{10}} = 0.$$

**Q:7** (10 + 5 points) (a) Let

$$f(z) = \frac{2}{1+z^2}$$

(i) Find the Taylor series of $f(z)$ in the region $|z-1| < \sqrt{2}$.
(ii) Find the Laurent series of $f(z)$ in the region $|z-1| > \sqrt{2}$.
(**Note:** Write the both series in their closed forms).

(i) $f(z) = \frac{1}{i}\left[\frac{1}{z-i} - \frac{1}{z+i}\right]$

$= \frac{1}{i}\left[\frac{1}{(z-1)+(1-i)} - \frac{1}{(z-1)+(1+i)}\right]$

$= \frac{1}{i}\frac{1}{(1-i)}\left[\frac{1}{1+\frac{z-1}{1-i}}\right] - \frac{1}{i(1+i)}\left[\frac{1}{1+\frac{z-1}{1+i}}\right], \quad \left|\frac{z-1}{1+i}\right| < 1 \Leftrightarrow |z-1| < \sqrt{2}$

$= \frac{1}{i(1-i)}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-1}{1-i}\right)^n - \frac{1}{i(1+i)}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-1}{1+i}\right)^n.$

(ii) $f(z) = \frac{1}{i}\left[\frac{1}{(z-1)\left[1+\frac{(1-i)}{z-1}\right]} - \frac{1}{(z-1)\left[1+\frac{1+i}{z-1}\right]}\right]$

$= \frac{1}{i(z-1)}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1-i}{z-1}\right)^n - \frac{1}{i(z-1)}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1+i}{z-1}\right)^n$

$= \frac{1}{i}\sum_{n=0}^{\infty}(-1)^n\frac{(1-i)^n}{(z-1)^{n+1}} - \frac{1}{i}\sum_{n=0}^{\infty}(-1)^n\frac{(1+i)^n}{(z-1)^{n+1}}$

$= \frac{1}{i}\sum_{n=0}^{\infty}(-1)^n\left[(1-i)^n - (1+i)^n\right](z-1)^{-n-1}$

(b) Find the radius of convergence of the power series $\sum_{k=1}^{\infty}\frac{k(k+1)}{2}z^{k-1}$.

$\lim_{n\to\infty}\left|\frac{(n+1)z^n(n+2)}{(n)z^{n-1}(n+1)}\right| = \lim_{n\to\infty}\left|\frac{n^2(1+\frac{1}{n})(1+\frac{2}{n})}{n^2(1+\frac{1}{n})}\cdot\frac{1}{z}\right|$

$= \left|\frac{1}{z}\right| < 1$

Radius of convergence is 1.

**Q:8** (15 points) Evaluate

$$\oint_{|z|=4} \frac{dz}{z^4 - 2z^3 - 3z^2}$$

by Cauchy's residue theorem.

Sol. $f(z) = \dfrac{1}{z^2(z^2-2z-3)} = \dfrac{1}{z^2(z+1)(z-3)}$

$f(z)$ has simple poles at $z=-1$ and $z=3$. It has a pole of order 2 at 0.

$$\text{Res}(f(z),0) = \lim_{z\to 0} \frac{d}{dz}\left[z^2 f(z)\right]$$

$$= \lim_{z\to 0} \frac{d}{dz}\left[\frac{1}{(z+1)(z-3)}\right]$$

$$= \lim_{z\to 0} \frac{-2z+2}{(z+1)^2(z-3)^2} = \frac{2}{9}$$

$$\text{Res}(f(z),-1) = \lim_{z\to -1}(z+1)f(z) = \lim_{z\to -1} \frac{1}{z^2(z-3)} = -\frac{1}{4}$$

$$\text{Res}(f(z),3) = \lim_{z\to 3}(z-3)f(z) = \lim_{z\to 3}\frac{1}{z^2(z+1)} = \frac{1}{36}$$

Inview of the Cauchy's residue theorem

$$\oint_{|z|=4} \frac{dz}{z^4-2z^3-3z^2} = 2\pi i\left[\frac{2}{9} - \frac{1}{4} + \frac{1}{36}\right]$$

$$= 2\pi i\,\frac{8-9+1}{36} = 0.$$

**Q:9** (15 points) Evaluate
$$\int_0^{2\pi} \frac{\sin^2(\theta)}{5+4\cos(\theta)}\,d\theta.$$

Sol: Using the substitutions
$$d\theta = \frac{dz}{iz}, \quad \cos\theta = \frac{z+z^{-1}}{2}, \quad \sin\theta = \frac{z-z^{-1}}{2i},$$
where $c$ is the unit circle directed counterclockwise

$$\int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta}\,d\theta = \oint_c \frac{\left(\frac{z-z^{-1}}{2i}\right)^2}{5+4\left(\frac{z+z^{-1}}{2}\right)} \cdot \frac{dz}{iz}$$
$$= \frac{i}{4}\oint_c \frac{(z^2-1)^2}{z^2(z+2)(2z+1)}\,dz$$
$$= \frac{i}{4}\,2\pi i\left[\mathrm{Res}(f;0) + \mathrm{Res}(f,-\tfrac{1}{2})\right]$$

where $f(z) = (z^2-1)^2 / z^2(z+2)(2z+1)$

$z=0$ is a Pole of order 2
$$\mathrm{Res}(f,0) = \lim_{z\to 0} \frac{d}{dz}\left[\frac{(z^2-1)^2}{(z+2)(2z+1)}\right]$$
$$= \lim_{z\to 0} \frac{(2z^2+5z+2)\cdot 2(z^2-1)\cdot 2z - (z^2-1)^2\cdot(4z+5)}{(2z^2+5z+2)^2}$$
$$= -\frac{5}{4}$$

$z=-\frac{1}{2}$ is a simple pole,
$$\mathrm{Res}(f;-\tfrac{1}{2}) = \lim_{z\to -\frac{1}{2}} (z+\tfrac{1}{2})\frac{(z^2-1)^2}{2z^2(z+2)(z+\frac{1}{2})}$$
$$= \lim_{z\to -\frac{1}{2}} \frac{(z^2-1)^2}{2z^2(z+2)}$$
$$= \frac{3}{4}$$

$$\therefore \int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta}\,d\theta = \frac{i}{4}\cdot 2\pi i\left[-\frac{5}{4}+\frac{3}{4}\right]$$
$$= -\frac{\pi}{2}\left(-\frac{1}{2}\right) = \frac{\pi}{4}$$