KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS

Department of Mathematics and Statistics

Math 302 Final Exam

Semester (212)

May 18, 2022 at 08:00-10:30 AM

Name: ....KEY........

I.D: .... Serial #: .... Sec: ....

*

Instructions

1. No electronic device (such as calculator, mobile phone, smart watch) is allowed in this exam.
2. Justify your answers. No credit is given for (correct) answers not supported by work.

Question	Points
1	/21
2	/21
3	/21
4	/21
5	/21
Total	/105

Question 1 (6+15 points)

I) Consider the matrix $\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ 5 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}$. The eigenvalue associated to the eigenvector $B = \begin{bmatrix} -1 \\ -i \\ 1 \end{bmatrix}$ is equal to

a) $2+i$ b) $2-i$ c) $2i$ d) $-2i$ e) $1-2i$

II) Evaluate the Cauchy principal value of the improper integral

$$\int_0^\infty \frac{x \sin x}{x^4+5x^2+4}\,dx.$$

I $AB = \begin{bmatrix} -2+i \\ -1-2i \\ 2-i \end{bmatrix} = (2-i)\begin{bmatrix} -1 \\ -i \\ 1 \end{bmatrix} \Rightarrow \lambda = 2-i$ 6

II $\displaystyle\int_{-\infty}^{\infty} \frac{x e^{ix}}{(x^2+1)(x^2+4)}\,dx = 2\pi i\left[\mathrm{Res}(f,i) + \mathrm{Res}(f,2i)\right]$ 5

$\mathrm{Res}(f,i) = \dfrac{i e^{i^2}}{4i^3+10i} = \dfrac{i e^{-1}}{6i} = \dfrac{e^{-1}}{6}$

$\mathrm{Res}(f,2i) = \dfrac{2i e^{2i^2}}{4(2i)^3+10(2i)} = \dfrac{2i e^{-2}}{-12i} = -\dfrac{e^{-2}}{6}$ 5

$\Rightarrow \mathrm{P.V}\displaystyle\int_{-\infty}^{\infty} \frac{x \sin x}{x^4+5x^2+4}\,dx = \mathrm{Im}\left(\int_{-\infty}^{\infty} \frac{x e^{ix}}{x^4+5x^2+4}\,dx\right)$

$= 2\pi\left(\dfrac{e^{-1}}{6} - \dfrac{e^{-2}}{6}\right)$

$= \dfrac{\pi}{3}\left(e^{-1}-e^{-2}\right)$ 5

$\Rightarrow \mathrm{P.V}\displaystyle\int_{0}^{\infty} \frac{x \sin x}{x^4+5x^2+4}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x \sin x}{x^4+5x^2+4}\,dx = \frac{\pi}{6}\left(e^{-1}-e^{-2}\right)$

Question 2 (6+15 points)

I) The set of points in the complex plane that satisfies $z^2 + \bar{z}^2 = 2$ is described to be:

a) parabola b) straight line c) circle d) ellipse e) hyperbola

---

II) Expand $f(z) = \frac{1}{z(z-3)}$ in a Laurent series valid for $1 < |z+1| < 4$.

(Show at least the terms with coefficients: $a_{-2}, a_{-1}, a_0, a_1, a_2$)

I $z^2 + \bar{z}^2 = (x^2 + 2ixy - y^2) + (x^2 - 2ixy - y^2) = 2$

$\Rightarrow 2x^2 - 2y^2 = 2 \Rightarrow x^2 - y^2 = 1$ (Eq. of hyperbola)

6

II $f(z) = \frac{1}{z(z-3)} = \frac{1}{3}\left[\frac{1}{z-3} - \frac{1}{z}\right]$ (partial fraction)

3

$f_1(z) = \frac{1}{z-3} = \frac{1}{-4+z+1} = -\frac{1}{4}\,\frac{1}{1-\frac{z+1}{4}} = -\frac{1}{4}\left[1 + \frac{z+1}{4} + \frac{(z+1)^2}{4^2} + \cdots\right]$

4

$f_2(z) = \frac{1}{z} = \frac{1}{z+1-1} = \frac{1}{z+1} \cdot \frac{1}{1-\frac{1}{z+1}} = \frac{1}{z+1}\left[1 + \frac{1}{z+1} + \frac{1}{(z+1)^2} + \cdots\right]$

4

f_1 valid for $|z+1| < 4$ and f_2 valid for $|z+1| > 1$

Thus $f(z) = \frac{1}{3}\left[\cdots - \frac{1}{(z+1)^2} - \frac{1}{z+1} - \frac{1}{4} - \frac{z+1}{4^2} - \frac{(z+1)^2}{4^3} + \cdots\right]$

4

valid for $1 < |z+1| < 4$.

Scanned with CamScanner

Question 3 (6+15 points)

I) If $f(z) = 3x - y + 5 + i(kx + 3y - 3)$ is analytic. Then $k =$

a) 3 b) 2 c) 1 d) -3 e) -1

II) Let $f(z) = \cot \pi z$ and C is the contour define by the rectangle bounded by $x = \frac{1}{2}$, $x = \pi$, $y = -1$ and $y = 1$.

Use the residue theorem to evaluate $\oint_C f(z)dz$.

(I) $u_x = 3 = v_y$ and $u_y = -1 = -v_x = -k \Rightarrow K = 1$ 6

(II) $f(z) = \cot \pi z = \dfrac{\cos \pi z}{\sin \pi z}$

poles of order 1 are $0, \mp 1, \mp 2, \ldots$ because $\pi z = n\pi$
$\Rightarrow z = n$ 5

inside C, we have only 1, 2 and 3

thus $\oint_C \cot \pi z \overset{dz}{=} 2\pi i \left[\text{Res}(f,1) + \text{Res}(f,2) + \text{Res}(f,3) \right]$ 5

$\text{Res}(f,1) = \dfrac{\cos \pi(1)}{\pi \cos \pi(1)} = \dfrac{1}{\pi}$

$\text{Res}(f,2) = \dfrac{\cos \pi(2)}{\pi \cos \pi(2)} = \dfrac{1}{\pi}$

$\text{Res}(f,3) = \dfrac{\cos \pi(3)}{\pi \cos \pi(3)} = \dfrac{1}{\pi}$

5

$\Rightarrow \oint_C \cos \pi z \, dz = 2\pi i \left[\frac{1}{\pi} + \frac{1}{\pi} + \frac{1}{\pi} \right] = 6i$

Question 4 (6+15 points)

I) If C is the line segment from the point $z = 1$ to the point $z = i$ and

$\int_C (z^2 - z + 2)dz = a + bi$, then $3(a+b) = \cdots$

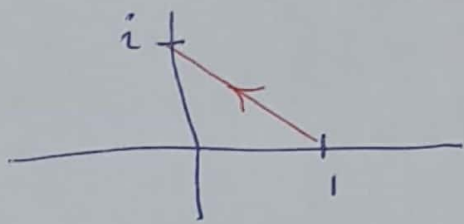
a) 1 b) 2 c) 3 d) -3 e) -1

II) Find the residue at the pole of the function $f(z) = \frac{1}{z \sin z}$.

I $C: \quad y = -x + 1$

$Z = x + i(-x+1)$

$dZ = (1-i)\,dx$



$$\int_C (z^2 - z + 2)\,dz = (1-i)\int_1^0 \left[x^2 - (1-x)^2 - x + 2 + (3x - 2x^2 - 1)i\right]dx$$

$$= -\frac{4}{3} + \frac{5}{3}i \implies 3(a+b) = 1$$

6

II $f_1(z) = z \sin z = z^2 - \frac{z^4}{3!} + \frac{z^6}{5!} - \frac{z^8}{7!} + \cdots$

$$= z^2\left[1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots\right]$$

5

$f_1(z)$ has a zero of order two

$\Rightarrow f = \frac{1}{z \sin z}$ has $z = 0$ a pole of order 2

2

$$\operatorname{Res}(f, 0) = \frac{1}{1!}\lim_{z\to 0}\frac{d}{dz}\left[z^2 \cdot \frac{1}{z\sin z}\right]$$ 3

$$= \lim_{z\to 0}\frac{\sin z - z\cos z}{\sin^2 z} = \lim_{z\to 0}\frac{\cos z + z\sin z - \cos z}{2\sin z\cos z}$$

$$= \lim_{z\to 0}\frac{z}{2\cos z} = 0$$

5

Scanned with CamScanner

Question 5 (6+15 points)

I) If C is the circle $|z+2i|=4$ then $\oint_C \frac{1}{z^2+9}dz = \cdots$

a) $\frac{\pi}{3}$ b) $-\frac{\pi}{3}$ c) $\frac{\pi}{3}i$ d) $-\frac{\pi}{3}i$ e) $\frac{2\pi}{3}$

II) Evaluate the real integral $\int_0^{2\pi} \frac{1}{1+3\cos^2\theta}d\theta$.

I $\oint_C \frac{1}{z^2+9}dz = 2\pi i \operatorname{Res}(f,-3i) = 2\pi i \frac{1}{2(-3i)} = -\frac{\pi}{3}$ 6

II By Substitution $z = e^{i\theta}$, $\cos\theta = \frac{1}{2}(z+z^{-1})$

$$\int_0^{2\pi} \frac{1}{1+3\cos^2\theta}d\theta = \frac{4}{i}\oint_C \frac{z}{3z^4+10z^2+3}dz$$, C is the unit circle 4

$$=\left(\frac{4}{i}\right)2\pi i\left[\operatorname{Res}\left(f,\frac{i}{\sqrt{3}}\right), \operatorname{Res}\left(f,\frac{-i}{\sqrt{3}}\right)\right]$$ 4

$$* \operatorname{Res}\left(f,\frac{i}{\sqrt{3}}\right) = \frac{z}{12z^3+20z}\Big|_{i/\sqrt{3}} = \frac{\frac{i}{\sqrt{3}}}{\frac{16i}{\sqrt{3}}} = \frac{1}{16}$$

$$* \operatorname{Res}\left(f,\frac{-i}{\sqrt{3}}\right) = \frac{\frac{-i}{\sqrt{3}}}{-\frac{16i}{\sqrt{3}}} = \frac{1}{16}$$ 4

$$\Rightarrow \int_0^{2\pi} \frac{1}{1+3\cos^2\theta}d\theta = \frac{4}{i}(2\pi i)\cdot\frac{2}{16} = \pi$$ 3