

Math 323 (Term 212)
Midterm Exam (Duration = 120 minutes)

Exercise 1 (6 points). Answer the following questions and justify your answers.

- (a) Is $\left(\frac{\mathbb{Z}}{6\mathbb{Z}}\right)^\times$, under multiplication, a group?
 - (b) Is the group $(\mathbb{Q}, +)$ cyclic?
 - (c) Is there an integer $n \geq 3$ such that the symmetric group S_n is abelian?
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Exercise 2 (6 points). Let G be a group of order $n = pqr$, where $p \leq q \leq r$ are prime numbers. Let H be a proper subgroup of G such that $\exists a, b \in H \setminus \{1\}$ with $|a| \neq |b|$. Find, all possible values for $|H|$.

Exercise 3 (6 points). Let $G := \frac{\mathbb{Z}}{n\mathbb{Z}}$, where $n = pqr$ and $2 \leq p \leq q \leq r$ are prime numbers. Let $H = \langle \bar{m} \rangle$ be a subgroup of G of order pq such that $1 < m \leq n - r$. Find the largest possible value for m .

Exercise 4 (6 points). Let S_{12} denote the symmetric group of degree 12 and let $\sigma \in S_{12}$ with $\sigma = \alpha_1 \alpha_2 \cdots \alpha_k$ where the α_i 's are disjoint r_i -cycles such that $r_i \geq 4$, for each i , and $r_1 + r_2 + \cdots + r_k = 12$. Determine all possible values for the order of σ and, for each case, state whether σ is even or odd.

Exercise 5 (6 points). Draw the lattice of all subgroups of $\frac{\mathbb{Z}}{66\mathbb{Z}}$ with all possible inclusions between subgroups and, for each subgroup, give its generator and order.

Exercise 6 (10 points). Let $G = \frac{\mathbb{Z}}{p^2\mathbb{Z}} \times \frac{\mathbb{Z}}{p\mathbb{Z}}$, where p is a prime number. Find the number of elements of G of order p .

Exercise 7 (10 points). Let $G = U(66)$ be the multiplication group modulo 66.

- (a) Determine the isomorphism class of G .
- (b) Express G as an internal direct product of cyclic groups.

[In G , we have: $5^2, 5^3, 5^4, 5^5, 5^6, 5^7, 5^8, 5^9, 5^{10} = 25, 59, 31, 23, 49, 47, 37, 53, 1$, respectively ; $17^2, 17^3, 17^4, 17^5, 17^6, 17^7, 17^8, 17^9, 17^{10} = 25, 29, 31, 65, 49, 41, 37, 35, 1$, respectively ; $23^2 = 43^2 = 65^2 = 1$.]

Exercise 8 (10 points). Show that \mathbb{Q} , under addition, has no proper subgroup of finite index

Exercise 9 (20 points). Let G be a non-trivial finite group. Consider the equivalence relation R defined on G by xRy if $y = gxg^{-1}$ for some $g \in G$. For each $x \in G$, let $\text{cl}(x)$ denote the equivalence class of x and let $\text{C}(x)$ denote the centralizer of x . That is, $\text{cl}(x) := \{gxg^{-1} \mid g \in G\}$ and $\text{C}(x) := \{g \in G \mid gx = xg\}$.

- (a) Prove that $|\text{cl}(x)| = |G : \text{C}(x)|$ for each $x \in G$.
- (b) Show that $|G| = \sum_{x \in G} |G : \text{C}(x)|$, where the sum runs over one representative x of $\text{cl}(x)$.
- (c) Show that $x \in Z(G)$ if and only if $|G : \text{C}(x)| = 1$.
- (d) Assume $|G| = p^n$, where p is a prime number and $n \geq 1$. Prove that $Z(G)$ is NOT trivial.
- (e) Deduce from (d) that if $|G| = p^2$, then G is abelian.