KFUPM - Department of Mathematics
MATH 323, Term 222
Exam I (Out of 80), Duration: 120 minutes

NAME: ID:

Solve the following Exercises.

Exercise 1: (Exercise 16, page 55) (16 points 4-3-3-6). Let $G = \{5, 15, 25, 35\}$.
(1) Prove that G is a group under multiplication modulo 40.
(2) What is the identity of G? What are the inverses of $5, 15$ and 35?
(3) Find the order of each non trivial element of G
(4) Form an isomorphism between G and $U(8)$.
Exercise 2: (Exercise 20, page 87, modified) (16 points: 5-5-6). Let G be an abelian group of order 55.
(1) What are the maximun and minimum numbers of possible elements of order 5?
(2) What are the maximun and minimum numbers of possible elements of order 11?
(3) If H is an abelian group of order 33. Is H a cysclic group? Justify.
Exercise 3 (16 points 4-4-4-4). Let G be a cyclic group with order $n = 16$.
(1) Prove that G cannot have more that 8 elements x such that $x^8 = e$.
(2) Prove that G cannot have more that 4 elements y such that $y^4 = e$.
(3) Find a noncyclic group H of order 4 which has 3 elements of order 2 but no element of order 4.
(4) Find an infinite abelian multiplicative group which has exactly two elements of order 4.
Exercise 4 (16 points 4-4-4-4). Let E be a finite set and $S(E)$ be the group of permutations of E.
(1) Prove that every $\sigma \in S(E)$ is a product of cycles. (**Theorem 5.1, page 98)**
(2) Assume that $|E| \geq 3$ and let A_n the subgroup of even permutations of E. Prove that for every distinct transpositions σ_1, σ_2 of E, $\sigma_1\sigma_2$ can be written as a 3-cycle or product of 3-cycles.
(3) Prove that every $\sigma \in A_n$ is a 3-cycle or a product of 3-cycles.
(4) Application: Set $E = \{1, 2, 3, 4, 5, 6\}$ and let σ the permutation given by $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 1 & 6 & 5 \end{pmatrix}$. Write σ as a product of cycles, transpositions and 3-cycles.
Exercise 5 (16 points 4-4-4-4). Let G be a group, g, h elements in G and T_g, T_h the inner automorphisms induced by g and h respectively.
(1) Prove that if $T_g = T_h$, then $h^{-1}g \in Z(G)$. (**Exercise 45, page 135**)
(2) Prove that if $h^{-1}g \in Z(G)$, then $T_g = T_h$. (**Exercise 47, page 135**)
(3) If G and H are isomorphic groups, are $Aut(G)$ and $Aut(H)$ isomorphic? Justify.
(4) If G and H are groups such that $Aut(G)$ is isomorphic to $Aut(H)$, are G and H isomorphic? Justify.