KFUPM - Department of Mathematics
MATH 323, Term 222
Final Exam (Out of 100), Duration: 180 minutes

NAME: ID:

Solve the following Exercises.

Exercise 1 (20 points, 4-4-4-4-4):
(1) Prove that $f = 2X^3 + X + 5$ is irreducible over $\mathbb{Z}$.
(2) Prove that $X^5 + 55X^3 - 222X^2 + 11X + 33$ is irreducible over $\mathbb{Q}$.
(4) Prove that $\mathbb{Q}[\sqrt{3}] = \{a + b\sqrt{3}|a, b \in \mathbb{Q}$ is a field.
(3) Prove that $\mathbb{Q}[X]/(X^2 - 3)$ is isomorphic to $\mathbb{Q}[\sqrt{3}]$ (Similar to **Exercise 46, page 286**).
(5) Prove that the ideal $(X^2 - 3)$ is a maximal ideal of $\mathbb{Q}[X]$.

Exercise 2: (similar to Example 8, page 318) (18 points, 6-4-4-2-2):
Let $D = \mathbb{Z}[\sqrt{-7}] = \{a + b\sqrt{-7}|a, b \in \mathbb{Z}\}$.
(1) Prove that $1 + \sqrt{-7}$ and $1 - \sqrt{-7}$ are irreducible but not prime.
(2) Prove that $6 + 2\sqrt{-7}$ and $1 + 3\sqrt{-7}$ are not associates (**Exercise 41, page 320**).
(3) Prove that 2 is irreducible in D.
(4) Find two factorizations of 8 in D.
(5) Is D a UFD (Unique Factorization Domain)? Justify.

Exercise 3 (16 points 5-5-6):
(1) Prove that $\mathbb{Q}(\sqrt{3})$ and $\mathbb{Q}(\sqrt{-3})$ are not isomorphic.
(2) Prove that the additive groups $(\mathbb{Z}[\sqrt{2}], +)$ and $(\mathbb{Z}[\sqrt{3}], +)$ are isomorphic.
(3) Prove that the rings $(\mathbb{Z}[\sqrt{2}], +, \times)$ and $(\mathbb{Z}[\sqrt{3}], +, \times)$ are not isomorphic.

Exercise 4: (Exercises 62-63-65-66-68 pages 246-247) (20 points, 4-4-4-4-4):
Let $\mathbb{F}$ be a field.
(1) Assume that $\mathbb{F}$ has exactly n elements. Prove that $x^{n-1} = 1$ for every nonzero x in $\mathbb{F}$.
(2) Assume that $\mathbb{F}$ has a prime characteristic p. Prove that $K = \{x \in \mathbb{F}|x^p = x\}$ is a subfield of $\mathbb{F}$.
(3) Assume that $p = 2$ and $\mathbb{F}$ has more than 2 elements. Prove that $(x+y)^3 \neq x^3+y^3$ for some x and y in $\mathbb{F}$.
(4) Assume that $\mathbb{F}$ has exactly 32 elements. Prove that the only subfields of $\mathbb{F}$ are $\{0, 1\}$ and $\mathbb{F}$.
(5) Assume that $p \neq 2$ and $\mathbb{F}^* = \mathbb{F} \setminus \{0\}$ is a cyclic multiplicative group. Prove that $\mathbb{F}$ is finite.

Exercise 5 (16 points 4-4-4-4): Let R be an integral domain and I and J two distinct ideals of R such that $R = I + J$.
(1) Prove that $IJ = I \cap J$.
(2) Let $\phi : R \longrightarrow R/I \times R/J$ be the ring homomorphism defined by $\phi(a) = (\bar{a}[I], \bar{a}[J])$ (where $\bar{a}[I]$ is the class of a mod I, $\bar{a}[J]$ is the class of a mod J). Prove

that $ker(\phi) = IJ$.
(3) Prove that ϕ is onto. [Hint, if $(\bar{x}[I], \bar{y}[J])$ is an element of $R/I \times R/J$, use $1 = a+b$ for some $a \in I$ and $b \in J$ to find an element $z \in R$ such that $\phi(z) = (\bar{x}[I], \bar{y}[J])$.
(4) Prove that R/IJ is isomorphic to $R/I \times R/J$. (Hint: Use the First Isomorphism Theorem applied to ϕ)

Exercise 6 (10 points, 5-5): Let R be a PID, $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \ldots$ be a chain of ideals of R and $I = \bigcup_{n \geq 1} I_n$.
(1) Prove that I is an ideal of R.
(2) Prove that there is there is a positive integer n_0 such that $I_n = I_{n_0}$ for all $n \geq n_0$.